

# The amplitude equation for weakly nonlinear reversible phase boundaries

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## Abstract

This technical note is a complement to an earlier paper [Benzoni-Gavage & Rosini, Comput. Math. Appl. 2009], which aims at a deeper understanding of a basic model for propagating phase boundaries that was proved to admit surface waves [Benzoni-Gavage, Nonlinear Anal. 1998]. The amplitude equation governing the evolution of weakly nonlinear surface waves for that model is computed explicitly, and is eventually found to have enough symmetry properties for the associated Cauchy problem to be locally well-posed.

## 1 Introduction

The reader is assumed to be familiar with the weakly nonlinear theory developed in [3], which is very much inspired from the seminal work [4]. In particular, we keep the same notation as in [3, Section 2], and consider the liquid-vapor phase transition model described in [3, Paragraphs 3.1, 3.2]. Our goal is to make explicit the amplitude equation, which corresponds to [3, Equation (2.20)] in an abstract framework. We adopt slightly different conventions compared with [3, Paragraph 3.3]. Namely, the eigenmodes with nonzero real part for the state ahead of the phase transition are<sup>1</sup>:

$$\beta_1^- := \frac{a_\ell - i u_\ell \eta_0}{c_\ell^2 - u_\ell^2}, \quad \beta_1^+ := \frac{-a_\ell - i u_\ell \eta_0}{c_\ell^2 - u_\ell^2} = -\overline{\beta_1^-}, \quad a_\ell := -c_\ell \sqrt{(c_\ell^2 - u_\ell^2) |\tilde{\eta}|^2 - \eta_0^2}. \quad (1)$$

The eigenmodes with nonzero real part for the state behind the phase transition are:

$$\beta_2^- := \frac{-a_r + i u_r \eta_0}{c_r^2 - u_r^2}, \quad \beta_2^+ := \frac{a_r + i u_r \eta_0}{c_r^2 - u_r^2} = -\overline{\beta_2^-}, \quad a_r := c_r \sqrt{(c_r^2 - u_r^2) |\tilde{\eta}|^2 - \eta_0^2}. \quad (2)$$

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<sup>1</sup>The main difference here with [3, Paragraph 3.3] is our sign convention for  $a_\ell$ , the latter being denoted by  $\alpha_\ell$  in [3, Paragraph 3.3].

The remaining, purely imaginary, eigenmodes are:

$$\beta_3^+ = \dots = \beta_{d+1}^+ := \frac{i\eta_0}{u_\ell}, \quad \beta_3^- = \dots = \beta_{d+1}^- := -\frac{i\eta_0}{u_r}.$$

The corresponding right eigenvectors are:

$$R_1^\pm := \begin{pmatrix} r_1^\pm \\ 0 \end{pmatrix}, \quad r_1^- := \begin{pmatrix} -i\eta_0 + u_\ell \beta_1^- \\ i c_\ell^2 \check{\eta} \\ -a_\ell \end{pmatrix}, \quad r_1^+ := \begin{pmatrix} i\eta_0 - u_\ell \beta_1^+ \\ -i c_\ell^2 \check{\eta} \\ -a_\ell \end{pmatrix} = \overline{r_1^-}, \quad (3)$$

$$R_2^\pm := \begin{pmatrix} 0 \\ r_2^\pm \end{pmatrix}, \quad r_2^- := \begin{pmatrix} -i\eta_0 - u_r \beta_2^- \\ i c_r^2 \check{\eta} \\ -a_r \end{pmatrix}, \quad r_2^+ := \begin{pmatrix} i\eta_0 + u_r \beta_2^+ \\ -i c_r^2 \check{\eta} \\ -a_r \end{pmatrix} = \overline{r_2^-}, \quad (4)$$

$$R_j^+ := \begin{pmatrix} r_j^+ \\ 0 \end{pmatrix}, \quad R_j^- := \begin{pmatrix} 0 \\ r_j^- \end{pmatrix}, \quad j = 3, \dots, d+1, \\ r_j^+ := \begin{pmatrix} 0 \\ \eta_0 \check{e}_{j-2} \\ u_\ell \check{\eta} \cdot \check{e}_{j-2} \end{pmatrix}, \quad r_j^- := \begin{pmatrix} 0 \\ \eta_0 \check{e}_{j-2} \\ u_r \check{\eta} \cdot \check{e}_{j-2} \end{pmatrix}, \quad (5)$$

with  $\check{e}_1 := \check{\eta}$  and the  $d-2$  vectors  $\check{e}_2, \dots, \check{e}_{d-1} \in \mathbb{R}^{d-1}$  span  $\check{\eta}^\perp$ .

The left eigenvectors are:

$$L_1^\pm := \begin{pmatrix} \ell_1^\pm \\ 0 \end{pmatrix}, \quad \ell_1^- := \frac{c_\ell^2 - u_\ell^2}{2 a_\ell (u_\ell a_\ell + i c_\ell^2 \eta_0)} \begin{pmatrix} i\eta_0 - 2 u_\ell \beta_1^+ \\ -i \check{\eta} \\ \beta_1^+ \end{pmatrix}, \\ \ell_1^+ := \frac{c_\ell^2 - u_\ell^2}{2 a_\ell (u_\ell a_\ell - i c_\ell^2 \eta_0)} \begin{pmatrix} -i\eta_0 + 2 u_\ell \beta_1^- \\ i \check{\eta} \\ -\beta_1^- \end{pmatrix} = \overline{\ell_1^-},$$

$$L_2^\pm := \begin{pmatrix} 0 \\ \ell_2^\pm \end{pmatrix}, \quad \ell_2^- := \frac{c_r^2 - u_r^2}{2 a_r (u_r a_r + i c_r^2 \eta_0)} \begin{pmatrix} -i\eta_0 - 2 u_r \beta_2^+ \\ i \check{\eta} \\ \beta_2^+ \end{pmatrix}, \\ \ell_2^+ := \frac{c_r^2 - u_r^2}{2 a_r (u_r a_r - i c_r^2 \eta_0)} \begin{pmatrix} i\eta_0 + 2 u_r \beta_2^- \\ -i \check{\eta} \\ -\beta_2^- \end{pmatrix} = \overline{\ell_2^-},$$

$$L_j^+ := \begin{pmatrix} \ell_j^+ \\ 0 \end{pmatrix}, \quad L_j^- := \begin{pmatrix} 0 \\ \ell_j^- \end{pmatrix}, \quad j = 3, \dots, d+1, \\ \ell_3^+ := \frac{1}{\eta_0^2 + u_\ell^2 |\check{\eta}|^2} \begin{pmatrix} u_\ell \\ -\eta_0 / (u_\ell |\check{\eta}|^2) \check{\eta} \\ -1 \end{pmatrix}, \quad \ell_3^- := \frac{1}{\eta_0^2 + u_r^2 |\check{\eta}|^2} \begin{pmatrix} -u_r \\ \eta_0 / (u_r |\check{\eta}|^2) \check{\eta} \\ 1 \end{pmatrix}, \\ \ell_j^+ := -\frac{1}{u_\ell \eta_0} \begin{pmatrix} 0 \\ \check{e}'_{j-2} \\ 0 \end{pmatrix}, \quad \ell_j^- := \frac{1}{u_r \eta_0} \begin{pmatrix} 0 \\ \check{e}'_{j-2} \\ 0 \end{pmatrix}, \quad j = 4, \dots, d+1,$$

where the  $d - 2$  vectors  $\check{e}'_2, \dots, \check{e}'_{d-1}$  belong to  $\check{\eta}^\perp$  and form the dual basis of  $\check{e}_2, \dots, \check{e}_{d-1}$ . Unlike the choice in [3, page 1476], the left eigenvectors here satisfy the normalization property:

$$(L_i^\pm)^* \check{\mathbb{A}}^d(\underline{v}) R_j^\pm = \delta_{i,j}, \quad (L_i^\pm)^* \check{\mathbb{A}}^d(\underline{v}) R_j^\mp = 0.$$

After linearizing the jump conditions we are left with the matrices

$$H(\underline{v}) := \begin{pmatrix} 0 & 0 & 1 & 0 & 0 & -1 \\ 0 & u_\ell I_{d-1} & 0 & 0 & -u_r I_{d-1} & 0 \\ c_\ell^2 - u_\ell^2 & 0 & 2u_\ell & -(c_r^2 - u_r^2) & 0 & -2u_r \\ (c_\ell^2 - u_\ell^2)u_\ell & 0 & u_\ell^2 + \mu & -(c_r^2 - u_r^2)u_r & 0 & -u_r^2 - \mu \end{pmatrix}, \quad (6)$$

where

$$\mu := \frac{1}{2} u_\ell^2 + g(\rho_\ell) = \frac{1}{2} u_r^2 + g(\rho_r), \quad (7)$$

and

$$J(\underline{v}) \eta := \begin{pmatrix} [\rho] \eta_0 \\ [p] \check{\eta} \\ 0 \\ (\mu [\rho] - [p]) \eta_0 \end{pmatrix}. \quad (8)$$

Here we have used the relation

$$\frac{1}{2} j[u] + [\rho f] = \mu [\rho] - [p],$$

where  $\mu$  is defined in (7), in order to simplify the last entry of  $J(\underline{v}) \eta$ .

The Lopatinskii determinant is defined by

$$\Delta(\eta) := \det \left( J(\underline{v}) \eta \quad H(\underline{v}) R_1^- \quad \dots \quad H(\underline{v}) R_{d+1}^- \right). \quad (9)$$

Each column vector in the above determinant is computed by using (8), (6) and (3), (4), (5) for the definition of the right eigenvectors. Then some simple manipulations on the rows and columns of the above determinant yield (see [1] for similar computations):

$$\Delta(\eta) = (-u_r \eta_0)^{d-2} u_r \det \begin{pmatrix} \check{e}_1 & \dots & \check{e}_{d-1} \end{pmatrix} \det \begin{pmatrix} [\rho] \eta_0 & a_\ell & a_r & |\check{\eta}|^2 \\ 0 & u_\ell a_\ell + i \eta_0 c_\ell^2 & u_r a_r + i \eta_0 c_r^2 & 2 u_r |\check{\eta}|^2 \\ -[p] \eta_0 & i \eta_0 u_\ell c_\ell^2 & i \eta_0 u_r c_r^2 & u_r^2 |\check{\eta}|^2 \\ [p] & -i u_\ell c_\ell^2 & -i u_r c_r^2 & \eta_0 \end{pmatrix}.$$

For future use, we introduce the quantity

$$\Upsilon := (-u_r \eta_0)^{d-2} u_r \det \begin{pmatrix} \check{e}_1 & \dots & \check{e}_{d-1} \end{pmatrix} \in \mathbb{R} \setminus \{0\}. \quad (10)$$

The expression of the Lopatinskii determinant then reduces to

$$\Delta(\eta) = -[\rho] [u] \Upsilon (\eta_0^2 + u_r^2 |\check{\eta}|^2) (u_\ell u_r a_\ell a_r + c_\ell^2 c_r^2 \eta_0^2). \quad (11)$$

We fix a root  $\underline{\eta}$  of the Lopatinskii determinant (see [1] for the properties of such roots, in particular the location of  $|\underline{\eta}_0|/|\underline{\eta}|$  with respect to various velocities associated with the phase transition). From now on, an underline refers to evaluation at the root  $\underline{\eta}$  of the Lopatinskii determinant.

We now define a vector  $\sigma \in \mathbb{C}^{d+2}$  by computing some of the minors of  $\Delta(\underline{\eta})$ . More precisely, the vectors  $H(\underline{v}) \underline{R}_1^-, \dots, H(\underline{v}) \underline{R}_{d+1}^-$  are linearly independent and we can thus define a vector  $\sigma \in \mathbb{C}^{d+2} \setminus \{0\}$  satisfying

$$\forall X \in \mathbb{C}^{d+2}, \quad \det \begin{pmatrix} X & H(\underline{v}) \underline{R}_1^- & \dots & H(\underline{v}) \underline{R}_{d+1}^- \end{pmatrix} = \sigma^* X.$$

This vector  $\sigma$  can be computed explicitly by performing some elementary manipulations on each minor of (9). We do not give the detailed calculations but rather give the expression of  $\sigma$ . We find

$$\sigma^* = \underline{\Upsilon} \begin{pmatrix} D_1 & \check{D} & \check{\eta}^T & D_{d+1} & D_{d+2} \end{pmatrix}, \quad (12)$$

where  $\underline{\Upsilon}$  denotes the quantity  $\Upsilon$  in (10) evaluated at the frequency  $\underline{\eta}$ , and<sup>2</sup>

$$\begin{aligned} D_1 + \mu D_{d+2} &:= -(\underline{\eta}_0^2 + u_r^2 |\check{\eta}|^2) ([u] c_\ell^2 c_r^2 \underline{\eta}_0 - i u_\ell u_r (c_r^2 \underline{a}_\ell - c_\ell^2 \underline{a}_r)), \\ \check{D} &:= -[u] u_r (\underline{a}_\ell (u_r \underline{a}_r - i c_r^2 \underline{\eta}_0) + \underline{a}_r (u_\ell \underline{a}_\ell - i c_\ell^2 \underline{\eta}_0)), \\ D_{d+1} &:= -i (\underline{\eta}_0^2 + u_r^2 |\check{\eta}|^2) (u_r c_r^2 \underline{a}_\ell - u_\ell c_\ell^2 \underline{a}_r), \\ D_{d+2} &:= [u] \underline{\eta}_0 (\underline{a}_\ell \underline{a}_r + c_\ell^2 c_r^2 |\check{\eta}|^2) \\ &\quad + i \underline{\eta}_0^2 (c_r^2 \underline{a}_\ell - c_\ell^2 \underline{a}_r) - i |\check{\eta}|^2 (u_r (u_\ell - 2 u_r) c_r^2 \underline{a}_\ell + u_\ell u_r c_\ell^2 \underline{a}_r). \end{aligned} \quad (13)$$

It remains to compute the coefficients  $\gamma_1, \gamma_2$  satisfying:

$$J(\underline{v}) \underline{\eta} + \gamma_1 H(\underline{v}) \underline{R}_1^- + \gamma_2 H(\underline{v}) \underline{R}_2^- = 0.$$

Observe that our convention differs from that in [3, page 1477]. We get:

$$\gamma_1 := \frac{[\rho] u_r \underline{\eta}_0}{u_r \underline{a}_\ell - i c_\ell^2 \underline{\eta}_0} = \frac{-i [\rho] c_r^2 \underline{\eta}_0^2}{\underline{a}_\ell (u_\ell \underline{a}_r - i c_r^2 \underline{\eta}_0)}, \quad \gamma_2 := \frac{-[\rho] u_\ell \underline{\eta}_0}{u_\ell \underline{a}_r - i c_r^2 \underline{\eta}_0} = \frac{i [\rho] c_\ell^2 \underline{\eta}_0^2}{\underline{a}_r (u_r \underline{a}_\ell - i c_\ell^2 \underline{\eta}_0)}, \quad (14)$$

where the equalities follow from the relation  $u_\ell u_r \underline{a}_\ell \underline{a}_r + c_\ell^2 c_r^2 \underline{\eta}_0^2 = 0$  that is satisfied by the root  $\underline{\eta}$  of the Lopatinskii determinant. For notational convenience, we also set

$$\gamma_3 = \dots = \gamma_{d+1} := 0.$$

Following [3, Proposition 2.2], the evolution of a weakly nonlinear phase transition is governed by a scalar amplitude  $w$  obeying a nonlocal Burgers equation:

$$a_0(k) \partial_\tau \widehat{w}(\tau, k) + \int_{\mathbb{R}} a_1(k - k', k') \widehat{w}(\tau, k - k') \widehat{w}(\tau, k') dk' = 0, \quad (15)$$

where  $a_0$  and  $a_1$  are given by Equations (2.24) and (2.25) in [3, page 1471]. With the present notation, this yields

$$a_0(k) = \begin{cases} \alpha_0 / (i k) & \text{if } k > 0, \\ \overline{\alpha_0} / (i k) & \text{if } k < 0, \end{cases}$$

and  $\alpha_0$  is a complex number whose definition is recalled in Equation (17) below. The expression of the kernel  $a_1$  is recalled and made explicit in Section 3 below.

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<sup>2</sup>Since  $\underline{a}_\ell$  is negative and  $\underline{a}_r$  is positive,  $D_{d+1}$  is nonzero and we can thus check that  $\sigma$  is a nonzero vector.

## 2 Computation of the coefficient $\alpha_0$

**Proposition 1.** *The coefficient  $\alpha_0$  in the expression of  $a_0$  is given by*

$$\alpha_0 = -\frac{[\rho] [u] \Upsilon}{\eta_0} (\eta_0^2 + u_r^2 |\check{\eta}|^2) \left\{ u_\ell^2 u_r^2 \left( \frac{a_\ell^2}{c_\ell^2} + \frac{a_r^2}{c_r^2} \right) + 2 c_\ell^2 c_r^2 \eta_0^2 \right\}, \quad (16)$$

and it coincides with the derivative of the Lopatinskii determinant  $\Delta$  with respect to  $\eta_0$  at its root  $\underline{\eta}$ . In particular,  $\alpha_0$  is a nonzero real number.

*Proof.* We recall that the expression of  $\alpha_0$  is

$$\alpha_0 = \sigma^* [\tilde{f}_0(\underline{v})] + i \sigma^* H(\underline{v}) \underline{R}_p^+ \frac{(\underline{L}_p^+)^* \gamma_q \underline{R}_q^-}{\underline{\beta}_p^+ - \underline{\beta}_q^-}, \quad (17)$$

where we use Einstein's summation convention over repeated indices. We first observe that the Hermitian product  $(\underline{L}_p^+)^* \underline{R}_q^-$ ,  $q = 1, 2$ , vanishes as soon as  $p$  is larger than 4. In the same way, the products  $(\underline{L}_1^+)^* \underline{R}_2^-$ ,  $(\underline{L}_3^+)^* \underline{R}_2^-$  and  $(\underline{L}_2^+)^* \underline{R}_1^-$  vanish so the expression of  $\alpha_0$  reduces to the sum of four terms:

$$\alpha_0 = \sigma^* [\tilde{f}_0(\underline{v})] + i \sigma^* H(\underline{v}) \underline{R}_2^+ \frac{(\underline{L}_2^+)^* \gamma_2 \underline{R}_2^-}{\underline{\beta}_2^+ - \underline{\beta}_2^-} + i \sigma^* H(\underline{v}) \underline{R}_1^+ \frac{(\underline{L}_1^+)^* \gamma_1 \underline{R}_1^-}{\underline{\beta}_1^+ - \underline{\beta}_1^-} + i \sigma^* H(\underline{v}) \underline{R}_3^+ \frac{(\underline{L}_3^+)^* \gamma_1 \underline{R}_1^-}{\underline{\beta}_3^+ - \underline{\beta}_1^-}. \quad (18)$$

We now compute each of these four quantities separately. Using

$$[\tilde{f}_0(\underline{v})] = \frac{1}{\eta_0} J(\underline{v}) \underline{\eta} - \frac{1}{\eta_0} \begin{pmatrix} 0 \\ [p] \check{\eta} \\ 0 \\ 0 \end{pmatrix},$$

and the orthogonality relation  $\sigma^* J(\underline{v}) \underline{\eta} = 0$ , we get

$$\begin{aligned} \sigma^* [\tilde{f}_0(\underline{v})] &= -[\rho] \Upsilon \frac{u_\ell u_r |\check{\eta}|^2}{\eta_0} \check{D} \\ &= [\rho] [u] \Upsilon \left\{ \frac{u_\ell u_r^2 |\check{\eta}|^2}{\eta_0} (\underline{a}_\ell (u_r \underline{a}_r - i c_r^2 \eta_0) + \underline{a}_r (u_\ell \underline{a}_\ell - i c_\ell^2 \eta_0)) \right\}. \end{aligned} \quad (19)$$

We now turn to the second term on the right in (18). There holds

$$\frac{(\underline{L}_2^+)^* \gamma_2 \underline{R}_2^-}{\underline{\beta}_2^+ - \underline{\beta}_2^-} = -\frac{c_r^2 |\check{\eta}|^2 (u_r \underline{a}_r - i c_r^2 \eta_0)}{2 \underline{a}_r^2 (\eta_0^2 + u_r^2 |\check{\eta}|^2)} \gamma_2,$$

and we also compute

$$\begin{aligned} i \sigma^* H(\underline{v}) \underline{R}_2^+ &= i \sigma^* \overline{H(\underline{v}) \underline{R}_2^-} = 2 \Upsilon c_r^2 \left\{ \eta_0 (D_{d+1} + u_r D_{d+2}) - u_r |\check{\eta}|^2 \check{D} \right\} \\ &= 2 [u] \Upsilon c_r^2 \underline{a}_r (\eta_0^2 + u_r^2 |\check{\eta}|^2) (u_r \underline{a}_\ell - i c_\ell^2 \eta_0), \end{aligned} \quad (20)$$

where the latter relation is obtained by using the expressions (13). Recalling the expression (14) of  $\gamma_2$ , we obtain

$$i \sigma^* H(\underline{v}) \underline{R}_2^+ \frac{(\underline{L}_2^+)^* \gamma_2 \underline{R}_2^-}{\underline{\beta}_2^+ - \underline{\beta}_2^-} = [\rho] [u] \Upsilon \left\{ i \frac{u_\ell u_r \underline{a}_\ell}{\underline{a}_r} c_r^2 |\check{\eta}|^2 (u_r \underline{a}_r - i c_r^2 \eta_0) \right\}. \quad (21)$$

We now examine the third term on the right in (18). There holds:

$$\frac{(\underline{L}_1^+)^* \gamma_1 \underline{R}_1^-}{\underline{\beta}_1^+ - \underline{\beta}_1^-} = -\frac{c_\ell^2 |\check{\eta}|^2 (u_\ell \underline{a}_\ell - i c_\ell^2 \eta_0)}{2 \underline{a}_\ell^2 (\eta_0^2 + u_\ell^2 |\check{\eta}|^2)} \gamma_1,$$

and we also compute

$$\begin{aligned} i \sigma^* H(\underline{v}) \underline{R}_1^+ &= i \sigma^* \overline{H(\underline{v}) \underline{R}_1^-} = 2 \Upsilon c_\ell^2 \left\{ u_\ell |\check{\eta}|^2 \check{D} - \eta_0 (D_{d+1} + u_\ell D_{d+2}) \right\} \\ &= -2 [u] \Upsilon c_\ell^2 \underline{a}_\ell (\eta_0^2 + u_r^2 |\check{\eta}|^2) (u_\ell \underline{a}_r - i c_r^2 \eta_0). \end{aligned} \quad (22)$$

Using the expression (14) of  $\gamma_1$ , we obtain

$$i \sigma^* H(\underline{v}) \underline{R}_1^+ \frac{(\underline{L}_1^+)^* \gamma_1 \underline{R}_1^-}{\underline{\beta}_1^+ - \underline{\beta}_1^-} = [\rho] [u] \Upsilon \left\{ i \frac{u_\ell u_r \underline{a}_r}{\underline{a}_\ell} c_\ell^2 |\check{\eta}|^2 \frac{\eta_0^2 + u_r^2 |\check{\eta}|^2}{\eta_0^2 + u_\ell^2 |\check{\eta}|^2} (u_\ell \underline{a}_\ell - i c_\ell^2 \eta_0) \right\}. \quad (23)$$

It remains to compute the last term on the right in (18) and to add the four expressions. First we compute

$$\frac{(\underline{L}_3^+)^* \gamma_1 \underline{R}_1^-}{\underline{\beta}_3^+ - \underline{\beta}_1^-} = -\frac{c_\ell^2}{\eta_0^2 + u_\ell^2 |\check{\eta}|^2} \gamma_1,$$

and we also compute

$$\sigma^* H(\underline{v}) \underline{R}_3^+ = -[u] \Upsilon u_\ell |\check{\eta}|^2 (2 D_{d+1} + (u_\ell + u_r) D_{d+2}), \quad (24)$$

where we have used the relation (which amounts to  $\sigma^* H(\underline{v}) \underline{R}_3^- = 0$ ):

$$D_1 + \eta_0 \check{D} + 2 u_r D_{d+1} + (\mu + u_r^2) D_{d+2} = 0.$$

The expression (24) can be factorized by using the definitions (13) of  $D_{d+1}$ ,  $D_{d+2}$ , and we obtain

$$\sigma^* H(\underline{v}) \underline{R}_3^+ = -\frac{[u]^2 \Upsilon u_\ell |\check{\eta}|^2}{\eta_0} (\eta_0^2 - u_\ell u_r |\check{\eta}|^2) (\underline{a}_r (u_r \underline{a}_\ell - i c_\ell^2 \eta_0) + \underline{a}_\ell (u_\ell \underline{a}_r - i c_r^2 \eta_0)). \quad (25)$$

Using (14), we derive the expression

$$i \sigma^* H(\underline{v}) \underline{R}_3^+ \frac{(\underline{L}_3^+)^* \gamma_1 \underline{R}_1^-}{\underline{\beta}_3^+ - \underline{\beta}_1^-} = [\rho] [u] \Upsilon \left\{ i [u] u_\ell c_\ell^2 |\check{\eta}|^2 \frac{\eta_0^2 - u_\ell u_r |\check{\eta}|^2}{\eta_0^2 + u_\ell^2 |\check{\eta}|^2} (u_r \underline{a}_r - i c_r^2 \eta_0) \right\}. \quad (26)$$

According to the decomposition (18), the coefficient  $\alpha_0$  is the sum of the four quantities in (19), (21), (23) and (26). Factorizing  $[\rho] [u] \Upsilon$  in each term, we first observe that the imaginary part of the sum equals zero. We can thus simplify  $\alpha_0$  by retaining only the real part of each term. This leads to the expression

$$\begin{aligned} \frac{\eta_0 \alpha_0}{[\rho] [u] \Upsilon} &= -c_\ell^2 c_r^2 \eta_0^2 (u_r^2 + u_\ell u_r) |\check{\eta}|^2 + u_\ell u_r \underline{a}_\ell \frac{c_r^2 \eta_0^2}{\underline{a}_r} c_r^2 |\check{\eta}|^2 \\ &\quad + u_\ell u_r \underline{a}_r \frac{c_\ell^2 \eta_0^2}{\underline{a}_\ell} c_\ell^2 |\check{\eta}|^2 \frac{\eta_0^2 + u_r^2 |\check{\eta}|^2}{\eta_0^2 + u_\ell^2 |\check{\eta}|^2} + (u_\ell u_r - u_\ell^2) |\check{\eta}|^2 c_\ell^2 c_r^2 \eta_0^2 \frac{\eta_0^2 - u_\ell u_r |\check{\eta}|^2}{\eta_0^2 + u_\ell^2 |\check{\eta}|^2}. \end{aligned}$$

At this stage, some elementary manipulations lead to the expression (16) of  $\alpha_0$ .

The link between  $\alpha_0$  and the partial derivative  $\partial_{\eta_0} \Delta(\underline{\eta})$  comes from the relation

$$\eta_0 \frac{\partial}{\partial \eta_0} (u_\ell u_r a_\ell a_r + c_\ell^2 c_r^2 \eta_0^2) \Big|_{\underline{\eta}} = u_\ell^2 u_r^2 \left( \frac{a_\ell^2}{c_\ell^2} + \frac{a_r^2}{c_r^2} \right) + 2 c_\ell^2 c_r^2 \eta_0^2,$$

which is obtained by differentiating (11) and the expressions (1), (2) with respect to  $\eta_0$  and then evaluating at  $\underline{\eta}$ .  $\square$

Since the coefficient  $\alpha_0$  is real, we obtain  $a_0(k) = \alpha_0/(ik)$  for all  $k \neq 0$ . In particular, the amplitude equation (15) reduces to

$$\partial_\tau \widehat{w}(\tau, k) + \frac{ik}{\alpha_0} \int_{\mathbb{R}} a_1(k - k', k') \widehat{w}(\tau, k - k') \widehat{w}(\tau, k') dk' = 0.$$

As far as smooth solutions are concerned, the Cauchy problem associated with this kind of nonlocal Burgers equation is known to be locally well-posed under rather simple algebraic conditions (see [2]). These conditions are invariant under multiplication by a nonzero real constant, so that it is sufficient to investigate whether they are satisfied by the slightly simpler kernel  $4\pi a_1$ . This is the purpose of the next section.

### 3 Computation of the quadratic kernel

We define the kernel  $q(k, k') := 4\pi a_1(k, k')$ . Following [3], we can decompose  $q$  as follows:

$$q(k, k') = \sum_{j=1}^5 q_j(k, k'),$$

where the kernels  $q_1, \dots, q_5$  are given by<sup>3</sup>

$$q_1(k, k') := \sigma(k + k')^* \sum_{j=0}^{d-1} \eta_j \left\{ d\tilde{f}^j(v_r) \cdot (\widehat{r}_+(k, 0) + \widehat{r}_+(k', 0)) - d\tilde{f}^j(v_\ell) \cdot (\widehat{r}_-(k, 0) + \widehat{r}_-(k', 0)) \right\}, \quad (27)$$

$$q_2(k, k') := -\sigma(k + k')^* \left\{ d^2 \tilde{f}^d(v_r) \cdot (\widehat{r}_+(k, 0), \widehat{r}_+(k', 0)) - d^2 \tilde{f}^d(v_\ell) \cdot (\widehat{r}_-(k, 0), \widehat{r}_-(k', 0)) \right\}, \quad (28)$$

$$q_3(k, k') := i(k + k') \int_0^{+\infty} L(k + k', z) d\mathbb{A}(v, \underline{\eta}) \cdot \widehat{r}(k, z) \cdot \widehat{r}(k', z) dz, \quad (29)$$

$$q_4(k, k') := \int_0^{+\infty} L(k + k', z) \frac{\partial}{\partial z} \left( d\check{\mathbb{A}}^d(v) \cdot \widehat{r}(k, z) \cdot \widehat{r}(k', z) \right) dz, \quad (30)$$

$$q_5(k, k') := - \int_0^{+\infty} L(k + k', z) \check{\mathbb{A}}(v, \underline{\eta}) \left( \frac{\partial \widehat{r}}{\partial z}(k, z) + \frac{\partial \widehat{r}}{\partial z}(k', z) \right) dz. \quad (31)$$

We first examine the kernel  $q_1$  and derive its expression for all values of  $(k, k')$ .

**Lemma 1.** *Let us define the quantity*

$$Q := 2[\rho][u] \Upsilon(\eta_0^2 + u_r^2 |\check{\eta}|^2) (\underline{\beta}_1^- + \underline{\beta}_2^-) i u_\ell u_r \underline{a}_\ell a_r \frac{u_\ell \underline{a}_r + i c_r^2 \eta_0}{u_\ell \underline{a}_r - i c_r^2 \eta_0}. \quad (32)$$

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<sup>3</sup>We keep the notation of [3] for the functions  $\widehat{r}_\pm, \widehat{r}$  and so on.

Then the kernel  $q_1$  in (27) satisfies

$$q_1(k, k') = \begin{cases} 0 & \text{if } k > 0 \text{ and } k' > 0, \\ \overline{Q} & \text{if } k > 0, k' < 0 \text{ and } k + k' > 0. \end{cases}$$

*Proof.* The function  $g^0$  is known to be an entropy for the isothermal Euler equations with corresponding flux  $(g^1, \dots, g^d)$ . We thus have the relations

$$\forall j = 0, \dots, d, \quad dg^0(v) A^j(v) = dg^j(v),$$

where we use the convention  $A^0(v) = I$  for all  $v$ . Using this relation between the Jacobian matrices, we get

$$\sum_{j=0}^{d-1} \eta_j d\tilde{f}^j(v_r) \cdot \underline{r}_2^- = \left( \frac{\sum_{j=0}^{d-1} \eta_j A^j(v_r) \underline{r}_2^-}{dg^0(v_r) \sum_{j=0}^{d-1} \eta_j A^j(v_r) \underline{r}_2^-} \right) = i \underline{\beta}_2^- \left( \frac{A^d(v_r) \underline{r}_2^-}{dg^d(v_r) \cdot \underline{r}_2^-} \right) = -i \underline{\beta}_2^- H(\underline{v}) \underline{R}_2^-.$$

Similarly we have

$$\sum_{j=0}^{d-1} \eta_j d\tilde{f}^j(v_\ell) \cdot \underline{r}_1^- = -i \underline{\beta}_1^- H(\underline{v}) \underline{R}_1^-.$$

For  $k > 0$  and  $k' > 0$ , we thus get

$$\begin{aligned} q_1(k, k') &= 2\sigma^* \sum_{j=0}^{d-1} \gamma_2 \eta_j d\tilde{f}^j(v_r) \cdot \underline{r}_2^- - 2\sigma^* \sum_{j=0}^{d-1} \gamma_1 \eta_j d\tilde{f}^j(v_\ell) \cdot \underline{r}_1^- \\ &= -2i\gamma_2 \underline{\beta}_2^- \sigma^* H(\underline{v}) \underline{R}_2^- + 2i\gamma_1 \underline{\beta}_1^- \sigma^* H(\underline{v}) \underline{R}_1^- = 0, \end{aligned}$$

because  $\sigma$  is orthogonal to both  $H(\underline{v}) \underline{R}_1^-$  and  $H(\underline{v}) \underline{R}_2^-$ .

Let us now consider the case  $k > 0, k' < 0$  and  $k + k' > 0$ . Using the same relations as above for the differentials  $dg^j$ , we obtain

$$\begin{aligned} q_1(k, k') &= \sigma^* \sum_{j=0}^{d-1} \eta_j d\tilde{f}^j(v_r) \cdot (\gamma_2 \underline{r}_2^- + \overline{\gamma}_2 \underline{r}_2^+) - \sigma^* \sum_{j=0}^{d-1} \eta_j d\tilde{f}^j(v_\ell) \cdot (\gamma_1 \underline{r}_1^- + \overline{\gamma}_1 \underline{r}_1^+) \\ &= -i\overline{\gamma}_2 \underline{\beta}_2^+ \sigma^* H(\underline{v}) \underline{R}_2^+ + i\overline{\gamma}_1 \underline{\beta}_1^+ \sigma^* H(\underline{v}) \underline{R}_1^+. \end{aligned}$$

The Hermitian products  $\sigma^* H(\underline{v}) \underline{R}_2^+$  and  $\sigma^* H(\underline{v}) \underline{R}_1^+$  have already been computed in the proof of Proposition 1, see (20) and (22). We then obtain

$$\begin{aligned} q_1(k, k') &= -2[u] \underline{\Upsilon} (\eta_0^2 + u_r^2 |\check{\eta}|^2) c_r^2 \underline{a}_r (u_r \underline{a}_\ell - i c_\ell^2 \eta_0) \overline{\gamma}_2 \underline{\beta}_2^+ \\ &\quad - 2[u] \underline{\Upsilon} (\eta_0^2 + u_r^2 |\check{\eta}|^2) c_\ell^2 \underline{a}_\ell (u_\ell \underline{a}_r - i c_r^2 \eta_0) \overline{\gamma}_1 \underline{\beta}_1^+. \end{aligned}$$

We use the definition (14) to obtain

$$c_r^2 \underline{a}_r (u_r \underline{a}_\ell - i c_\ell^2 \eta_0) \overline{\gamma}_2 = c_\ell^2 \underline{a}_\ell (u_\ell \underline{a}_r - i c_r^2 \eta_0) \overline{\gamma}_1 = i[\rho] c_\ell^2 c_r^2 \eta_0^2 \frac{u_\ell \underline{a}_r - i c_r^2 \eta_0}{u_\ell \underline{a}_r + i c_r^2 \eta_0},$$

and the claim follows using the relation  $c_\ell^2 c_r^2 \eta_0^2 = -u_\ell u_r \underline{a}_\ell \underline{a}_r$ .  $\square$



Deriving the expression of the kernels  $q_3, q_4, q_5$  requires the expression of the row vector  $L(k + k', z)$ , which we derive right now.

**Lemma 2.** *For  $k > 0$ , there holds*

$$L(k, z) = \left( \frac{\omega_1}{\gamma_1} \exp(-k \underline{\beta}_1^+ z) \tilde{\ell}_1 + \frac{\omega_3}{\gamma_1} \exp(-k \underline{\beta}_3^+ z) \tilde{\ell}_3 - \frac{\omega_2}{\gamma_2} \exp(-k \underline{\beta}_2^+ z) \tilde{\ell}_2 \right),$$

where we have set

$$\begin{aligned} \tilde{\ell}_1 &:= (i \eta_0 - 2 u_\ell \underline{\beta}_1^+ & -i \check{\eta}^T & \underline{\beta}_1^+) , & \tilde{\ell}_3 &:= (-u_\ell^2 |\check{\eta}|^2 & \eta_0 \check{\eta}^T & u_\ell |\check{\eta}|^2) , \\ \tilde{\ell}_2 &:= (-i \eta_0 - 2 u_r \underline{\beta}_2^+ & i \check{\eta}^T & \underline{\beta}_2^+) , \end{aligned}$$

and

$$\begin{aligned} \omega_1 &:= [\rho] [u] \underline{\Upsilon} \frac{\eta_0^2 + u_r^2 |\check{\eta}|^2}{\eta_0^2 + u_\ell^2 |\check{\eta}|^2} i u_\ell \eta_0 (u_r \underline{a}_r - i c_r^2 \eta_0), \\ \omega_3 &:= [\rho] [u]^2 \underline{\Upsilon} \frac{\eta_0^2 - u_\ell u_r |\check{\eta}|^2}{\eta_0^2 + u_\ell^2 |\check{\eta}|^2} (u_r \underline{a}_r - i c_r^2 \eta_0), & \omega_2 &:= [\rho] [u] \underline{\Upsilon} i u_r \eta_0 (u_\ell \underline{a}_\ell - i c_\ell^2 \eta_0). \end{aligned}$$

*Proof.* We first observe that  $\sigma$  is orthogonal to the vectors  $H(\underline{v}) \underline{R}_p^+$  for  $p \geq 4$ , so for  $k > 0$  the expression of  $L(k, z)$  reduces to

$$L(k, z) = \sum_{p=1}^3 \sigma^* H(\underline{v}) \underline{R}_p^+ \exp(-k \underline{\beta}_p^+ z) (\underline{L}_p^+)^*.$$

The expression of the products  $\sigma^* H(\underline{v}) \underline{R}_p^+$ ,  $p = 1, 2, 3$  can be found in (20), (22), (25), and we use the definitions of the left eigenvectors  $\underline{L}_p^+$  to derive the expressions given in Lemma 2.  $\square$

We now examine the kernel  $q_5$ , which, as  $q_1$  but unlike  $q_2, q_3, q_4$ , does not contain any term in  $p''(\rho_{\ell, r})$ .

**Lemma 3.** *With  $Q$  defined in (32), the kernel  $q_5$  in (31) satisfies*

$$q_5(k, k') = \begin{cases} 0 & \text{if } k > 0 \text{ and } k' > 0, \\ \overline{Q} \frac{k'}{k} & \text{if } k > 0, k' < 0 \text{ and } k + k' > 0. \end{cases}$$

*Proof.* For  $k > 0$  and  $k' > 0$ , we compute

$$\check{\mathbb{A}}(v, \underline{\eta}) \left( \frac{\partial \hat{r}}{\partial z}(k, z) + \frac{\partial \hat{r}}{\partial z}(k', z) \right) = \begin{pmatrix} i \gamma_1 (\underline{\beta}_1^-)^2 (k \exp(k \underline{\beta}_1^- z) + k' \exp(k' \underline{\beta}_1^- z)) A^d(v_\ell) \underline{r}_1^- \\ i \gamma_2 (\underline{\beta}_2^-)^2 (k \exp(k \underline{\beta}_2^- z) + k' \exp(k' \underline{\beta}_2^- z)) A^d(v_r) \underline{r}_2^- \end{pmatrix}.$$

Using the orthogonality properties  $\tilde{\ell}_1 A^d(v_\ell) \underline{r}_1^- = \tilde{\ell}_3 A^d(v_\ell) \underline{r}_1^- = \tilde{\ell}_3 A^d(v_r) \underline{r}_2^- = 0$ , we get  $q_5(k, k') = 0$  if  $k > 0$  and  $k' > 0$  because the integrand in (31) vanishes.

Let us now consider the case  $k > 0$ ,  $k' < 0$  and  $k + k' > 0$ . From the previous argument, we still find that the term  $L(k + k', z) \check{\mathbb{A}}(v, \underline{\eta}) \partial_z \hat{r}(k, z)$  vanishes. We thus get

$$\begin{aligned} q_5(k, k') &= - \int_0^{+\infty} L(k + k', z) \check{\mathbb{A}}(v, \underline{\eta}) \frac{\partial \hat{r}}{\partial z}(k', z) dz \\ &= -i k' \int_0^{+\infty} L(k + k', z) \begin{pmatrix} \overline{\gamma}_1 (\underline{\beta}_1^+)^2 \exp(k' \underline{\beta}_1^+ z) A^d(v_\ell) \underline{r}_1^+ \\ \overline{\gamma}_2 (\underline{\beta}_2^+)^2 \exp(k' \underline{\beta}_2^+ z) A^d(v_r) \underline{r}_2^+ \end{pmatrix} dz. \end{aligned}$$

We now use the expression of  $L(k+k', z)$  in Lemma 2. The expression of  $q_5(k, k')$  is simplified by recalling the orthogonality property  $\tilde{\ell}_3 A^d(v_\ell) \underline{r}_1^+ = 0$  and we get

$$\begin{aligned} q_5(k, k') &= -i k' \int_0^{+\infty} \frac{\overline{\gamma_1}}{\gamma_1} \omega_1 \tilde{\ell}_1 A^d(v_\ell) \underline{r}_1^+ (\underline{\beta}_1^+)^2 \exp(-k \underline{\beta}_1^+ z) dz \\ &\quad - i k' \int_0^{+\infty} \frac{\overline{\gamma_2}}{\gamma_2} \omega_2 \tilde{\ell}_2 A^d(v_r) \underline{r}_2^+ (\underline{\beta}_2^+)^2 \exp(-k \underline{\beta}_2^+ z) dz, \\ &= -i \frac{k'}{k} \left\{ \frac{\overline{\gamma_1}}{\gamma_1} \omega_1 \tilde{\ell}_1 A^d(v_\ell) \underline{r}_1^+ \underline{\beta}_1^+ + \frac{\overline{\gamma_2}}{\gamma_2} \omega_2 \tilde{\ell}_2 A^d(v_r) \underline{r}_2^+ \underline{\beta}_2^+ \right\}. \end{aligned}$$

The conclusion of Lemma 3 then follows from the relations

$$\begin{aligned} \frac{\overline{\gamma_1}}{\gamma_1} &= -\frac{\overline{\gamma_2}}{\gamma_2} = -\frac{u_\ell \underline{a}_r - i c_r^2 \underline{\eta}_0}{u_\ell \underline{a}_r + i c_r^2 \underline{\eta}_0}, \\ \omega_1 \tilde{\ell}_1 A^d(v_\ell) \underline{r}_1^+ &= -\omega_2 \tilde{\ell}_2 A^d(v_r) \underline{r}_2^+ = 2[\rho][u] \underline{\Upsilon} (\underline{\eta}_0^2 + u_r^2 |\underline{\check{J}}|^2) u_\ell u_r \underline{a}_\ell \underline{a}_r. \end{aligned}$$

□

We immediately get

**Corollary 1.** *With  $Q$  defined in (32), the kernels  $q_1, q_5$  in (27), (31) satisfy*

$$(q_1 + q_5)(k, k') = \begin{cases} 0 & \text{if } k > 0 \text{ and } k' > 0, \\ \overline{Q} \left(1 + \frac{k'}{k}\right) & \text{if } k > 0, k' < 0 \text{ and } k + k' > 0. \end{cases}$$

Our goal now is to derive an explicit expression for the kernels  $q_2, q_3, q_4$  in (28), (29), (30). This is more intricate because these kernels are quadratic with respect to the vector  $\hat{r}(k, z)$  and there is more algebra involved to obtain a factorized expression in each region of the  $(k, k')$ -plane. We begin with some preliminary computations that will be useful in Propositions 2 and 3 below.

**Lemma 4.** *The coordinates (13) of the vector  $\sigma$  satisfy*

$$\begin{aligned} \gamma_1 \check{D} &= -[\rho][u] u_r \underline{\eta}_0 (u_r \underline{a}_r - i c_r^2 \underline{\eta}_0), \\ \gamma_2 \check{D} &= [\rho][u] u_r \underline{\eta}_0 (u_\ell \underline{a}_\ell - i c_\ell^2 \underline{\eta}_0), \\ \gamma_1 D_{d+1} &= -[\rho] u_r (\underline{\eta}_0^2 + u_r^2 |\underline{\check{J}}|^2) (u_\ell \underline{a}_r + i c_r^2 \underline{\eta}_0), \\ \gamma_2 D_{d+1} &= -[\rho] u_\ell (\underline{\eta}_0^2 + u_r^2 |\underline{\check{J}}|^2) (u_r \underline{a}_\ell + i c_\ell^2 \underline{\eta}_0), \\ \gamma_1 D_{d+2} &= [\rho] (\underline{\eta}_0^2 + u_r^2 |\underline{\check{J}}|^2) (u_r \underline{a}_r + i c_r^2 \underline{\eta}_0) - [\rho][u] u_r |\underline{\check{J}}|^2 (u_r \underline{a}_r - i c_r^2 \underline{\eta}_0), \\ \gamma_2 D_{d+2} &= [\rho] (\underline{\eta}_0^2 + u_r^2 |\underline{\check{J}}|^2) (u_\ell \underline{a}_\ell + i c_\ell^2 \underline{\eta}_0) + [\rho][u] u_r |\underline{\check{J}}|^2 (u_\ell \underline{a}_\ell - i c_\ell^2 \underline{\eta}_0). \end{aligned}$$

*Proof.* From the definition (13), there holds

$$\check{D} = [u] u_r (\underline{a}_r (u_r \underline{a}_\ell - i c_\ell^2 \underline{\eta}_0) + \underline{a}_\ell (u_\ell \underline{a}_r - i c_r^2 \underline{\eta}_0)),$$

and we then use the expression (14) of  $\gamma_1, \gamma_2$  to compute  $\gamma_1 \check{D}$  and  $\gamma_2 \check{D}$ .

The expressions of  $\gamma_{1,2} D_{d+1}$  follow from the observation that  $D_{d+1}$  satisfies

$$\begin{aligned} \underline{\eta}_0 D_{d+1} &= -(\underline{\eta}_0^2 + u_r^2 |\underline{\check{J}}|^2) (u_r \underline{a}_\ell - i c_\ell^2 \underline{\eta}_0) (u_\ell \underline{a}_r + i c_r^2 \underline{\eta}_0) \\ &= (\underline{\eta}_0^2 + u_r^2 |\underline{\check{J}}|^2) (u_r \underline{a}_\ell + i c_\ell^2 \underline{\eta}_0) (u_\ell \underline{a}_r - i c_r^2 \underline{\eta}_0). \end{aligned}$$

We then use again (14).

To compute the product  $\gamma_1 D_{d+2}$ , we recall the relation (22) which we found in the proof of Proposition 1. It reads

$$\vartheta_0 (D_{d+1} + u_\ell D_{d+2}) - u_\ell |\check{\vartheta}|^2 \check{D} = [u] \underline{a}_\ell (\vartheta_0^2 + u_r^2 |\check{\vartheta}|^2) (u_\ell \underline{a}_r - i c_r^2 \vartheta_0).$$

We multiply the latter relation by  $\gamma_1$  and use the previous expressions of  $\gamma_1 \check{D}$  and  $\gamma_1 D_{d+1}$  to obtain that of  $\gamma_1 D_{d+2}$ . The expression of  $\gamma_2 D_{d+2}$  is then easily deduced by using  $\gamma_2/\gamma_1 = i c_\ell^2 \vartheta_0^2/(u_r \underline{a}_r)$ .  $\square$

We now compute the kernels  $q_2, q_3, q_4$  in the region  $\{k > 0, k' > 0\}$ .

**Proposition 2.** *Let us define the quantities*

$$\begin{aligned} Q_\ell &:= [\rho] [u] \underline{\Upsilon} u_\ell u_r \frac{\underline{a}_r}{\underline{a}_\ell} (\vartheta_0^2 + u_r^2 |\check{\vartheta}|^2) (\vartheta_0^2 + u_\ell^2 |\check{\vartheta}|^2) \gamma_1 (i \vartheta_0 - u_\ell \underline{\beta}_1^-), \\ Q_r &:= [\rho] [u] \underline{\Upsilon} u_\ell u_r \frac{\underline{a}_\ell}{\underline{a}_r} (\vartheta_0^2 + u_r^2 |\check{\vartheta}|^2)^2 \gamma_2 (i \vartheta_0 + u_r \underline{\beta}_2^-), \\ Q_\# &:= 2 [\rho] \underline{\Upsilon} (\vartheta_0^2 + u_r^2 |\check{\vartheta}|^2) (\vartheta_0^2 + u_\ell u_r |\check{\vartheta}|^2) i c_\ell^2 c_r^2 \vartheta_0 \left( \frac{c_r^2 \gamma_2}{\rho_r u_r} - \frac{c_\ell^2 \gamma_1}{\rho_\ell u_\ell} \right). \end{aligned} \quad (33)$$

Then the kernels  $q_2, q_3, q_4$  defined in (28), (29) and (30) satisfy

$$(q_2 + q_3 + q_4)(k, k') = \left( \frac{p''(\rho_\ell)}{2} + \frac{c_\ell^2}{\rho_\ell} \right) Q_\ell + \left( \frac{p''(\rho_r)}{2} + \frac{c_r^2}{\rho_r} \right) Q_r + Q_\#,$$

for all  $k > 0$  and  $k' > 0$ .

*Proof.* We first recall the expressions of the second differentials that are involved in the kernels  $q_2, q_3, q_4$ , see [3, page 1480]:

$$\sum_{k=1}^{d-1} \check{\vartheta}_k d^2 f^k(v_{\ell,r}) \cdot (v, v) = p''(\rho_{\ell,r}) \begin{pmatrix} 0 \\ \rho^2 \check{\vartheta} \\ 0 \end{pmatrix} + \frac{2}{\rho_{\ell,r}} \begin{pmatrix} 0 \\ \check{\vartheta} \cdot \check{j} \check{j} \\ \check{\vartheta} \cdot \check{j} (j_d - u_{\ell,r} \rho) \end{pmatrix}, \quad (34)$$

$$\begin{aligned} d^2 \tilde{f}^d(v_{\ell,r}) \cdot (v, v) &= \begin{pmatrix} d^2 f^d(v_{\ell,r}) \cdot (v, v) \\ d^2 g^d(v_{\ell,r}) \cdot (v, v) \end{pmatrix} = p''(\rho_{\ell,r}) \begin{pmatrix} 0 \\ 0 \\ \rho^2 \\ u_{\ell,r} \rho^2 \end{pmatrix} + \frac{2}{\rho_{\ell,r}} \begin{pmatrix} 0 \\ (j_d - u_{\ell,r} \rho) \check{j} \\ (j_d - u_{\ell,r} \rho)^2 \\ 0 \end{pmatrix} \\ &\quad + \frac{1}{\rho_{\ell,r}} \begin{pmatrix} 0 \\ 0 \\ 0 \\ 3 u_{\ell,r} (j_d - u_{\ell,r} \rho)^2 - u_{\ell,r} c_{\ell,r}^2 \rho^2 + 2 c_{\ell,r}^2 \rho j_d + u_{\ell,r} \check{j} \cdot \check{j} \end{pmatrix}. \end{aligned} \quad (35)$$

The proof of Proposition 2 splits in several steps. We assume from now on that  $k$  and  $k'$  are both positive and we wish to compute the expression of the kernel  $q_2 + q_3 + q_4$ .

• Step 1: computation of the  $p''(\rho_\ell)$  factor. In this first step, we collect all the terms that involve  $p''(\rho_\ell)$  in  $q_2 + q_3 + q_4$ . The contribution of the kernel  $q_2$  equals

$$\begin{aligned} &\underline{\Upsilon} (D_{d+1} + u_\ell D_{d+2}) \gamma_1^2 (i \vartheta_0 - u_\ell \underline{\beta}_1^-)^2 \\ &= -[\rho] [u] \underline{\Upsilon} \gamma_1 (i \vartheta_0 - u_\ell \underline{\beta}_1^-)^2 \left\{ (\vartheta_0^2 + u_r^2 |\check{\vartheta}|^2) i c_r^2 \vartheta_0 + (u_r \underline{a}_r - i c_r^2 \vartheta_0) u_\ell u_r |\check{\vartheta}|^2 \right\}, \end{aligned} \quad (36)$$

where we have used Lemma 4 to compute the product  $(D_{d+1} + u_\ell D_{d+2}) \gamma_1$ .

The contribution of the kernel  $q_3$  equals

$$\begin{aligned} & i(k+k') \int_0^{+\infty} \frac{\omega_1}{\gamma_1} (-i \check{\eta}^T) e^{-(k+k') \underline{\beta}_1^+ z} \gamma_1^2 (i \eta_0 - u_\ell \underline{\beta}_1^-)^2 \check{\eta} e^{(k+k') \underline{\beta}_1^- z} dz \\ & + i(k+k') \int_0^{+\infty} \frac{\omega_3}{\gamma_1} (\eta_0 \check{\eta}^T) e^{-(k+k') \underline{\beta}_3^+ z} \gamma_1^2 (i \eta_0 - u_\ell \underline{\beta}_1^-)^2 \check{\eta} e^{(k+k') \underline{\beta}_1^- z} dz \\ & = \gamma_1 (i \eta_0 - u_\ell \underline{\beta}_1^-)^2 \left\{ \omega_1 \frac{|\check{\eta}|^2}{\underline{\beta}_1^+ - \underline{\beta}_1^-} + \omega_3 |\check{\eta}|^2 \frac{i \eta_0}{\underline{\beta}_3^+ - \underline{\beta}_1^-} \right\}. \end{aligned}$$

Similarly, the contribution of the kernel  $q_4$  reads

$$\begin{aligned} & - \int_0^{+\infty} \frac{\omega_1}{\gamma_1} \underline{\beta}_1^+ e^{-(k+k') \underline{\beta}_1^+ z} \gamma_1^2 (i \eta_0 - u_\ell \underline{\beta}_1^-)^2 (k+k') \underline{\beta}_1^- e^{(k+k') \underline{\beta}_1^- z} dz \\ & - \int_0^{+\infty} \frac{\omega_3}{\gamma_1} u_\ell |\check{\eta}|^2 e^{-(k+k') \underline{\beta}_3^+ z} \gamma_1^2 (i \eta_0 - u_\ell \underline{\beta}_1^-)^2 (k+k') \underline{\beta}_1^- e^{(k+k') \underline{\beta}_1^- z} dz \\ & = - \gamma_1 (i \eta_0 - u_\ell \underline{\beta}_1^-)^2 \left\{ \omega_1 \frac{\underline{\beta}_1^+ \underline{\beta}_1^-}{\underline{\beta}_1^+ - \underline{\beta}_1^-} + \omega_3 |\check{\eta}|^2 \frac{u_\ell \underline{\beta}_1^-}{\underline{\beta}_3^+ - \underline{\beta}_1^-} \right\}. \end{aligned}$$

Adding the contributions of  $q_3$  and  $q_4$  gives the term

$$\gamma_1 (i \eta_0 - u_\ell \underline{\beta}_1^-)^2 \left\{ -\frac{1}{2 \underline{a}_\ell} \left( \frac{\underline{a}_\ell^2}{c_\ell^2} + c_\ell^2 |\check{\eta}|^2 \right) \omega_1 + u_\ell |\check{\eta}|^2 \omega_3 \right\}. \quad (37)$$

We now use the definitions of  $\omega_1$  and  $\omega_3$ , see Lemma 2, and add the contributions in (36) and (37) in order to obtain the  $p''(\rho_\ell)$  term in  $q_2 + q_3 + q_4$ . We first obtain that the sum of the right hand side of (36) and the expression in (37) equals

$$-[\rho][u] \Upsilon \gamma_1 (i \eta_0 - u_\ell \underline{\beta}_1^-)^2 \frac{\eta_0^2 + u_r^2 |\check{\eta}|^2}{2 c_\ell^2 (\eta_0^2 + u_\ell^2 |\check{\eta}|^2)} (u_\ell \underline{a}_\ell + i c_\ell^2 \eta_0) \left( c_r^2 \eta_0^2 + \frac{u_r \underline{a}_r}{u_\ell \underline{a}_\ell} u_\ell^2 c_\ell^2 |\check{\eta}|^2 \right).$$

This last expression is simplified a little further by observing that we have

$$(i \eta_0 - u_\ell \underline{\beta}_1^-) (u_\ell \underline{a}_\ell + i c_\ell^2 \eta_0) = \frac{-1}{c_\ell^2 - u_\ell^2} (u_\ell^2 \underline{a}_\ell^2 + c_\ell^4 \eta_0^2) = -c_\ell^2 (\eta_0^2 + u_\ell^2 |\check{\eta}|^2),$$

and

$$c_r^2 \eta_0^2 + \frac{u_r \underline{a}_r}{u_\ell \underline{a}_\ell} u_\ell^2 c_\ell^2 |\check{\eta}|^2 = u_\ell u_r \frac{\underline{a}_r}{\underline{a}_\ell} \left( c_\ell^2 |\check{\eta}|^2 - \frac{\underline{a}_\ell^2}{c_\ell^2} \right) = u_\ell u_r \frac{\underline{a}_r}{\underline{a}_\ell} (\eta_0^2 + u_\ell^2 |\check{\eta}|^2).$$

Eventually, we find that the sum of all the terms that involve  $p''(\rho_\ell)$  factorizes as  $Q_\ell/2$  where  $Q_\ell$  is defined in (33).

• Step 2: computation of the  $p''(\rho_r)$  factor. We follow the same strategy as in the first step and compute the contribution that involves  $p''(\rho_r)$  in each kernel. The contribution of the kernel  $q_2$  equals

$$-\Upsilon (D_{d+1} + u_r D_{d+2}) \gamma_2^2 (i \eta_0 + u_r \underline{\beta}_2^-)^2 = [\rho][u] \Upsilon \gamma_2 (i \eta_0 + u_r \underline{\beta}_2^-)^2 \frac{u_\ell \underline{a}_\ell}{c_r^2} \left\{ i \eta_0 u_r \underline{a}_r - u_r^2 c_r^2 |\check{\eta}|^2 \right\}, \quad (38)$$

where we have used Lemma 4 to simplify  $(D_{d+1} + u_r D_{d+2}) \gamma_2$ . The contribution of the kernel  $q_3$  equals

$$\begin{aligned} i(k+k') \int_0^{+\infty} \frac{\omega_2}{\gamma_2} (i \check{\eta}^T) e^{-(k+k') \beta_2^+ z} \gamma_2^2 (i \eta_0 + u_r \beta_2^-)^2 \check{\eta} e^{(k+k') \beta_2^- z} dz \\ = -\gamma_2 (i \eta_0 + u_r \beta_2^-)^2 \omega_2 \frac{|\check{\eta}|^2}{\beta_2^+ - \beta_2^-}, \end{aligned}$$

and the contribution of the kernel  $q_4$  reads

$$\begin{aligned} \int_0^{+\infty} \frac{\omega_2}{\gamma_2} \beta_2^+ e^{-(k+k') \beta_2^+ z} \gamma_2^2 (i \eta_0 + u_r \beta_2^-)^2 (k+k') \beta_2^- e^{(k+k') \beta_2^- z} dz \\ = \gamma_2 (i \eta_0 + u_r \beta_2^-)^2 \omega_2 \frac{\beta_2^+ \beta_2^-}{\beta_2^+ - \beta_2^-}. \end{aligned}$$

Adding the contributions of  $q_3$  and  $q_4$  gives the term

$$-\gamma_2 (i \eta_0 + u_r \beta_2^-)^2 \frac{\omega_2}{2 \underline{a}_r} \left( \frac{\underline{a}_r^2}{c_r^2} + c_r^2 |\check{\eta}|^2 \right).$$

When we add the latter term with the expression in (38), we obtain

$$-[\rho][u] \Upsilon \gamma_2 (i \eta_0 + u_r \beta_2^-)^2 (\eta_0^2 + u_r^2 |\check{\eta}|^2) \frac{i u_r \eta_0}{2 \underline{a}_r} (u_\ell \underline{a}_\ell + i c_\ell^2 \eta_0),$$

and this quantity is further simplified by using the relation

$$(i \eta_0 + u_r \beta_2^-) i \eta_0 (u_\ell \underline{a}_\ell + i c_\ell^2 \eta_0) = -u_\ell \underline{a}_\ell \frac{u_r^2 \underline{a}_r^2 + c_r^4 \eta_0^2}{c_r^2 (c_r^2 - u_r^2)} = -u_\ell \underline{a}_\ell (\eta_0^2 + u_r^2 |\check{\eta}|^2).$$

Eventually, we find that the sum of all the terms that involve  $p''(\rho_r)$  factorizes as  $Q_r/2$ .

• Step 3: computation of the remaining terms. In order to prove Proposition 2, we can assume from now on, and without loss of generality that  $p''(\rho_\ell) = p''(\rho_r) = 0$  in (34) and (35). With this simplification, we compute

$$\sum_{k=1}^{d-1} \check{\eta}_k d^2 f^k(v_\ell) \cdot (x_1^-, x_1^-) = -\frac{2 i c_\ell^4 |\check{\eta}|^2}{\rho_\ell} \begin{pmatrix} 0 \\ -i \check{\eta} \\ \beta_1^- \end{pmatrix}, \quad (39)$$

$$d^2 \tilde{f}^d(v_\ell) \cdot (x_1^-, x_1^-) = \frac{2 c_\ell^2}{\rho_\ell} \begin{pmatrix} 0 \\ -i c_\ell^2 \beta_1^- \check{\eta} \\ c_\ell^2 |\check{\eta}|^2 + (i \eta_0 - u_\ell \beta_1^-)^2 \\ i c_\ell^2 \eta_0 \beta_1^- + u_\ell (i \eta_0 - u_\ell \beta_1^-)^2 \end{pmatrix}, \quad (40)$$

$$\sum_{k=1}^{d-1} \check{\eta}_k d^2 f^k(v_r) \cdot (x_2^-, x_2^-) = \frac{2 i c_r^4 |\check{\eta}|^2}{\rho_r} \begin{pmatrix} 0 \\ i \check{\eta} \\ \beta_2^- \end{pmatrix}, \quad (41)$$

$$d^2 \tilde{f}^d(v_r) \cdot (x_2^-, x_2^-) = \frac{2 c_r^2}{\rho_r} \begin{pmatrix} 0 \\ i c_r^2 \beta_2^- \check{\eta} \\ c_r^2 |\check{\eta}|^2 + (i \eta_0 + u_r \beta_2^-)^2 \\ -i c_r^2 \eta_0 \beta_2^- + u_r (i \eta_0 + u_r \beta_2^-)^2 \end{pmatrix}, \quad (42)$$

where in (40) and (42), we have used the relations

$$c_\ell^2 (\underline{\beta}_1^-)^2 = c_\ell^2 |\check{\mathcal{J}}|^2 + (i \mathcal{J}_0 - u_\ell \underline{\beta}_1^-)^2, \quad c_r^2 (\underline{\beta}_2^-)^2 = c_r^2 |\check{\mathcal{J}}|^2 + (i \mathcal{J}_0 + u_r \underline{\beta}_2^-)^2.$$

With these expressions, let us look first at the kernel  $q_2$  in (28). Using (40), we compute

$$\begin{aligned} \sigma^* d^2 \tilde{f}^d(v_\ell) \cdot (x_1^-, x_1^-) \gamma_1^2 &= \frac{2c_\ell^2}{\rho_\ell} \underline{\Upsilon} \gamma_1 \left\{ i c_\ell^2 \underline{\beta}_1^- (\mathcal{J}_0 \gamma_1 D_{d+2} - |\check{\mathcal{J}}|^2 \gamma_1 \check{D}) + c_\ell^2 |\check{\mathcal{J}}|^2 \gamma_1 D_{d+1} \right. \\ &\quad \left. + (i \mathcal{J}_0 - u_\ell \underline{\beta}_1^-)^2 \gamma_1 (D_{d+1} + u_\ell D_{d+2}) \right\}, \end{aligned}$$

and Lemma 4 turns this expression into

$$\begin{aligned} \sigma^* d^2 \tilde{f}^d(v_\ell) \cdot (x_1^-, x_1^-) \gamma_1^2 &= \frac{2c_\ell^2}{\rho_\ell} [\rho] \underline{\Upsilon} (\mathcal{J}_0^2 + u_r^2 |\check{\mathcal{J}}|^2) \gamma_1 \left\{ u_r \underline{a}_r (u_\ell \underline{a}_\ell + i c_\ell^2 \mathcal{J}_0) \underline{\beta}_1^- - u_r c_\ell^2 |\check{\mathcal{J}}|^2 (u_\ell \underline{a}_r + i c_r^2 \mathcal{J}_0) \right\} \\ &\quad - \frac{2c_\ell^2}{\rho_\ell} [\rho] [u] \underline{\Upsilon} \gamma_1 (i \mathcal{J}_0 - u_\ell \underline{\beta}_1^-)^2 \left\{ (\mathcal{J}_0^2 + u_r^2 |\check{\mathcal{J}}|^2) i c_r^2 \mathcal{J}_0 + u_\ell u_r |\check{\mathcal{J}}|^2 (u_r \underline{a}_r - i c_r^2 \mathcal{J}_0) \right\}. \quad (43) \end{aligned}$$

Similarly, we derive the relation

$$\begin{aligned} \sigma^* d^2 \tilde{f}^d(v_r) \cdot (x_2^-, x_2^-) \gamma_2^2 &= -\frac{2c_r^2}{\rho_r} [\rho] \underline{\Upsilon} (\mathcal{J}_0^2 + u_r^2 |\check{\mathcal{J}}|^2) \gamma_2 \left\{ u_\ell \underline{a}_\ell (u_r \underline{a}_r + i c_r^2 \mathcal{J}_0) \underline{\beta}_2^- + u_\ell c_r^2 |\check{\mathcal{J}}|^2 (u_r \underline{a}_\ell + i c_\ell^2 \mathcal{J}_0) \right\} \\ &\quad + \frac{2c_r^2}{\rho_r} [\rho] [u] \underline{\Upsilon} \gamma_2 (i \mathcal{J}_0 + u_r \underline{\beta}_2^-)^2 \left\{ (\mathcal{J}_0^2 + u_r^2 |\check{\mathcal{J}}|^2) i c_\ell^2 \mathcal{J}_0 + u_r^2 |\check{\mathcal{J}}|^2 (u_\ell \underline{a}_\ell - i c_\ell^2 \mathcal{J}_0) \right\}. \quad (44) \end{aligned}$$

The kernels  $q_3, q_4$  both read as a sum of two contributions, one from the system ahead of the phase boundary, and one from the system behind the phase boundary. We compute each of these contributions separately in order to combine them with either (43) or (44). Using (39), the ‘left’ contribution of  $q_3$  equals

$$\begin{aligned} &i(k+k') \int_0^{+\infty} \frac{\omega_1}{\gamma_1} \tilde{\ell}_1 e^{-(k+k') \underline{\beta}_1^+ z} \gamma_1^2 \frac{-2i c_\ell^4 |\check{\mathcal{J}}|^2}{\rho_\ell} \begin{pmatrix} 0 \\ -i \check{\mathcal{J}} \\ \underline{\beta}_1^- \end{pmatrix} e^{(k+k') \underline{\beta}_1^- z} dz \\ &+ i(k+k') \int_0^{+\infty} \frac{\omega_3}{\gamma_1} \tilde{\ell}_3 e^{-(k+k') \underline{\beta}_3^+ z} \gamma_1^2 \frac{-2i c_\ell^4 |\check{\mathcal{J}}|^2}{\rho_\ell} \begin{pmatrix} 0 \\ -i \check{\mathcal{J}} \\ \underline{\beta}_1^- \end{pmatrix} e^{(k+k') \underline{\beta}_1^- z} dz \\ &= \frac{2c_\ell^4}{\rho_\ell} \gamma_1 \omega_1 \frac{|\check{\mathcal{J}}|^2}{\underline{\beta}_1^+ - \underline{\beta}_1^-} \tilde{\ell}_1 \begin{pmatrix} 0 \\ -i \check{\mathcal{J}} \\ \underline{\beta}_1^- \end{pmatrix} + \frac{2c_\ell^4}{\rho_\ell} \gamma_1 \omega_3 \frac{|\check{\mathcal{J}}|^2}{\underline{\beta}_3^+ - \underline{\beta}_1^-} \tilde{\ell}_3 \begin{pmatrix} 0 \\ -i \check{\mathcal{J}} \\ \underline{\beta}_1^- \end{pmatrix}, \quad (45) \end{aligned}$$

and, similarly (using now (41) rather than (39)), the ‘right’ contribution of  $q_3$  equals

$$i(k+k') \int_0^{+\infty} \frac{\omega_2}{\gamma_2} \tilde{\ell}_2 e^{-(k+k') \underline{\beta}_2^+ z} \gamma_2^2 \frac{2i c_r^4 |\check{\eta}|^2}{\rho_r} \begin{pmatrix} 0 \\ i \check{\eta} \\ \underline{\beta}_2^- \end{pmatrix} e^{(k+k') \underline{\beta}_2^- z} dz$$

$$= -\frac{2c_r^4}{\rho_r} \gamma_2 \omega_2 \frac{|\check{\eta}|^2}{\underline{\beta}_2^+ - \underline{\beta}_2^-} \tilde{\ell}_2 \begin{pmatrix} 0 \\ i \check{\eta} \\ \underline{\beta}_2^- \end{pmatrix}. \quad (46)$$

The kernel  $q_4$  is computed by first observing that the vectors  $d^2 f^d(v_\ell) \cdot (r_1^-, r_1^-)$ , and  $d^2 f^d(v_r) \cdot (r_2^-, r_2^-)$  are obtained by retaining only the three first coordinates in (40) and (42):

$$d^2 f^d(v_\ell) \cdot (r_1^-, r_1^-) = \frac{2c_\ell^4 \underline{\beta}_1^-}{\rho_\ell} \begin{pmatrix} 0 \\ -i \check{\eta} \\ \underline{\beta}_1^- \end{pmatrix}, \quad d^2 f^d(v_r) \cdot (r_2^-, r_2^-) = \frac{2c_r^4 \underline{\beta}_2^-}{\rho_r} \begin{pmatrix} 0 \\ i \check{\eta} \\ \underline{\beta}_2^- \end{pmatrix}.$$

Consequently, the ‘left’ contribution of  $q_4$  equals

$$- \int_0^{+\infty} \frac{\omega_1}{\gamma_1} \tilde{\ell}_1 e^{-(k+k') \underline{\beta}_1^+ z} \gamma_1^2 \frac{2c_\ell^4 \underline{\beta}_1^-}{\rho_\ell} \begin{pmatrix} 0 \\ -i \check{\eta} \\ \underline{\beta}_1^- \end{pmatrix} (k+k') \underline{\beta}_1^- e^{(k+k') \underline{\beta}_1^- z} dz$$

$$- \int_0^{+\infty} \frac{\omega_3}{\gamma_1} \tilde{\ell}_1 e^{-(k+k') \underline{\beta}_3^+ z} \gamma_1^2 \frac{2c_\ell^4 \underline{\beta}_1^-}{\rho_\ell} \begin{pmatrix} 0 \\ -i \check{\eta} \\ \underline{\beta}_1^- \end{pmatrix} (k+k') \underline{\beta}_1^- e^{(k+k') \underline{\beta}_1^- z} dz$$

$$= -\frac{2c_\ell^4}{\rho_\ell} \gamma_1 \omega_1 \frac{(\underline{\beta}_1^-)^2}{\underline{\beta}_1^+ - \underline{\beta}_1^-} \tilde{\ell}_1 \begin{pmatrix} 0 \\ -i \check{\eta} \\ \underline{\beta}_1^- \end{pmatrix} - \frac{2c_\ell^4}{\rho_\ell} \gamma_1 \omega_3 \frac{(\underline{\beta}_1^-)^2}{\underline{\beta}_3^+ - \underline{\beta}_1^-} \tilde{\ell}_3 \begin{pmatrix} 0 \\ -i \check{\eta} \\ \underline{\beta}_1^- \end{pmatrix}, \quad (47)$$

and the ‘right’ contribution of  $q_4$  equals

$$\frac{2c_r^4}{\rho_r} \gamma_2 \omega_2 \frac{(\underline{\beta}_2^-)^2}{\underline{\beta}_2^+ - \underline{\beta}_2^-} \tilde{\ell}_2 \begin{pmatrix} 0 \\ i \check{\eta} \\ \underline{\beta}_2^- \end{pmatrix}. \quad (48)$$

The ‘left’ contribution of  $q_3 + q_4$  is obtained by adding the expressions in (45) and (47), which gives

$$-\frac{2c_\ell^4}{\rho_\ell} \gamma_1 \omega_1 \frac{(\underline{\beta}_1^-)^2 - |\check{\eta}|^2}{\underline{\beta}_1^+ - \underline{\beta}_1^-} \tilde{\ell}_1 \begin{pmatrix} 0 \\ -i \check{\eta} \\ \underline{\beta}_1^- \end{pmatrix} - \frac{2c_\ell^4}{\rho_\ell} \gamma_1 \omega_3 \frac{(\underline{\beta}_1^-)^2 - |\check{\eta}|^2}{\underline{\beta}_3^+ - \underline{\beta}_1^-} \tilde{\ell}_3 \begin{pmatrix} 0 \\ -i \check{\eta} \\ \underline{\beta}_1^- \end{pmatrix}$$

$$= -\frac{2c_\ell^2}{\rho_\ell} \gamma_1 (i \eta_0 - u_\ell \underline{\beta}_1^-)^2 \left\{ \omega_1 \tilde{\ell}_1 \begin{pmatrix} 0 \\ -i \check{\eta} \\ \underline{\beta}_1^- \end{pmatrix} + \omega_3 \tilde{\ell}_3 \begin{pmatrix} 0 \\ -i \check{\eta} \\ \underline{\beta}_1^- \end{pmatrix} \right\}$$

$$= \frac{c_\ell^2}{\rho_\ell \underline{a}_\ell} \gamma_1 (i \eta_0 - u_\ell \underline{\beta}_1^-)^2 \left\{ (\eta_0^2 + u_\ell^2 |\check{\eta}|^2 - 2c_\ell^2 |\check{\eta}|^2) \omega_1 + 2u_\ell \underline{a}_\ell |\check{\eta}|^2 \omega_3 \right\}. \quad (49)$$

The ‘right’ contribution of  $q_3 + q_4$  is obtained by adding the expressions in (46) and (48), which gives

$$\begin{aligned} \frac{2c_r^4}{\rho_r} \gamma_2 \omega_2 \frac{(\beta_2^-)^2 - |\check{\eta}|^2}{\beta_2^+ - \beta_2^-} \tilde{\ell}_2 \begin{pmatrix} 0 \\ i \check{\eta} \\ \beta_2^- \end{pmatrix} &= \frac{2c_r^2}{\rho_r} \gamma_2 \omega_2 (i \eta_0 + u_r \beta_2^-)^2 \frac{\beta_2^+ \beta_2^- - |\check{\eta}|^2}{\beta_2^+ - \beta_2^-} \\ &= \frac{c_r^2}{\rho_r \underline{a}_r} \gamma_2 (i \eta_0 + u_r \beta_2^-)^2 (\eta_0^2 + u_r^2 |\check{\eta}|^2 - 2c_r^2 |\check{\eta}|^2) \omega_2. \quad (50) \end{aligned}$$

We can now compute the ‘left’ contribution of the full kernel  $q_2 + q_3 + q_4$  by combining the expression in (43) with the one in (49). We use here the expressions of  $\omega_1$  and  $\omega_3$  given in Lemma 2. The sum of (43) and (49) reads

$$\begin{aligned} A_\ell := & \frac{2c_\ell^2}{\rho_\ell} [\rho] \Upsilon (\eta_0^2 + u_r^2 |\check{\eta}|^2) \gamma_1 \left\{ u_r \underline{a}_r (u_\ell \underline{a}_\ell + i c_\ell^2 \eta_0) \beta_1^- - u_r c_\ell^2 |\check{\eta}|^2 (u_\ell \underline{a}_r + i c_r^2 \eta_0) \right\} \\ & - \frac{2c_\ell^2}{\rho_\ell} [\rho] [u] \Upsilon \gamma_1 (i \eta_0 - u_\ell \beta_1^-)^2 \left\{ (\eta_0^2 + u_r^2 |\check{\eta}|^2) i c_r^2 \eta_0 + u_\ell u_r |\check{\eta}|^2 (u_r \underline{a}_r - i c_r^2 \eta_0) \right\} \\ & + \frac{c_\ell^2}{\rho_\ell \underline{a}_\ell} [\rho] [u] \Upsilon \gamma_1 (i \eta_0 - u_\ell \beta_1^-)^2 \left\{ 2 u_\ell \underline{a}_\ell |\check{\eta}|^2 [u] \frac{\eta_0^2 - u_\ell u_r |\check{\eta}|^2}{\eta_0^2 + u_\ell^2 |\check{\eta}|^2} (u_r \underline{a}_r - i c_r^2 \eta_0) \right. \\ & \left. + (\eta_0^2 + u_\ell^2 |\check{\eta}|^2 - 2c_\ell^2 |\check{\eta}|^2) i u_\ell \eta_0 (u_r \underline{a}_r - i c_r^2 \eta_0) \frac{\eta_0^2 + u_r^2 |\check{\eta}|^2}{\eta_0^2 + u_\ell^2 |\check{\eta}|^2} \right\}. \end{aligned}$$

The three last rows in the definition of  $A_\ell$  are factorized after a little bit of algebra, and we get

$$\begin{aligned} A_\ell = & \frac{2c_\ell^2}{\rho_\ell} [\rho] \Upsilon (\eta_0^2 + u_r^2 |\check{\eta}|^2) \gamma_1 \left\{ u_r \underline{a}_r (u_\ell \underline{a}_\ell + i c_\ell^2 \eta_0) \beta_1^- - u_r c_\ell^2 |\check{\eta}|^2 (u_\ell \underline{a}_r + i c_r^2 \eta_0) \right\} \\ & - \frac{1}{\rho_\ell} [\rho] [u] \Upsilon \gamma_1 (i \eta_0 - u_\ell \beta_1^-)^2 u_\ell u_r \frac{\underline{a}_r}{\underline{a}_\ell} (u_\ell \underline{a}_\ell + i c_\ell^2 \eta_0) (\eta_0^2 + u_r^2 |\check{\eta}|^2). \end{aligned}$$

We have thus shown that the ‘left’ contribution  $A_\ell$  of  $q_2 + q_3 + q_4$  reads

$$A_\ell = \frac{2c_\ell^2}{\rho_\ell} [\rho] \Upsilon (\eta_0^2 + u_r^2 |\check{\eta}|^2) \gamma_1 \left\{ u_r \underline{a}_r (u_\ell \underline{a}_\ell + i c_\ell^2 \eta_0) \beta_1^- - u_r c_\ell^2 |\check{\eta}|^2 (u_\ell \underline{a}_r + i c_r^2 \eta_0) \right\} + \frac{c_\ell^2}{\rho_\ell} Q_\ell. \quad (51)$$

The ‘right’ contribution of  $q_2 + q_3 + q_4$  by combining the expression in (44) (with a minus sign, recall the definition (28) of the kernel  $q_2$ ) with the one in (50):

$$\begin{aligned} A_r := & \frac{2c_r^2}{\rho_r} [\rho] \Upsilon (\eta_0^2 + u_r^2 |\check{\eta}|^2) \gamma_2 \left\{ u_\ell \underline{a}_\ell (u_r \underline{a}_r + i c_r^2 \eta_0) \beta_2^- + u_\ell c_r^2 |\check{\eta}|^2 (u_r \underline{a}_\ell + i c_\ell^2 \eta_0) \right\} \\ & - \frac{2c_r^2}{\rho_r} [\rho] [u] \Upsilon \gamma_2 (i \eta_0 + u_r \beta_2^-)^2 \left\{ (\eta_0^2 + u_r^2 |\check{\eta}|^2) i c_\ell^2 \eta_0 + u_r^2 |\check{\eta}|^2 (u_\ell \underline{a}_\ell - i c_\ell^2 \eta_0) \right\} \\ & + \frac{c_r^2}{\rho_r \underline{a}_r} \gamma_2 (i \eta_0 + u_r \beta_2^-)^2 (\eta_0^2 + u_r^2 |\check{\eta}|^2 - 2c_r^2 |\check{\eta}|^2) \omega_2. \end{aligned}$$

Once again, the last two rows are factorized after a few calculations that we skip, and we get

$$\begin{aligned} A_r = & \frac{2c_r^2}{\rho_r} [\rho] \Upsilon (\eta_0^2 + u_r^2 |\check{\eta}|^2) \gamma_2 \left\{ u_\ell \underline{a}_\ell (u_r \underline{a}_r + i c_r^2 \eta_0) \beta_2^- + u_\ell c_r^2 |\check{\eta}|^2 (u_r \underline{a}_\ell + i c_\ell^2 \eta_0) \right\} \\ & - \frac{1}{\rho_r} [\rho] [u] \Upsilon \gamma_2 (i \eta_0 + u_r \beta_2^-)^2 u_\ell u_r \frac{\underline{a}_\ell}{\underline{a}_r} (u_r \underline{a}_r + i c_r^2 \eta_0) (\eta_0^2 + u_r^2 |\check{\eta}|^2). \end{aligned}$$



We have thus shown that the ‘left’ contribution  $A_\ell$  of  $q_2 + q_3 + q_4$  reads

$$A_r = \frac{2c_r^2}{\rho_r} [\rho] \underline{\Upsilon} (\eta_0^2 + u_r^2 |\check{\eta}|^2) \gamma_2 \left\{ u_\ell \underline{a}_\ell (u_r \underline{a}_r + i c_r^2 \eta_0) \underline{\beta}_2^- + u_\ell c_r^2 |\check{\eta}|^2 (u_r \underline{a}_\ell + i c_\ell^2 \eta_0) \right\} + \frac{c_r^2}{\rho_r} Q_r. \quad (52)$$

The final step of the proof consists in simplifying the remaining terms in  $A_\ell$  and  $A_r$ . More specifically, the first factor in the expression (51) of  $A_\ell$  can be simplified as follows:

$$\begin{aligned} & u_\ell \left\{ u_r \underline{a}_r (u_\ell \underline{a}_\ell + i c_\ell^2 \eta_0) \underline{\beta}_1^- - u_r c_\ell^2 |\check{\eta}|^2 (u_\ell \underline{a}_r + i c_r^2 \eta_0) \right\} \\ &= u_r \underline{a}_r (u_\ell \underline{a}_\ell + i c_\ell^2 \eta_0) (u_\ell \underline{\beta}_1^- - i \eta_0) - c_\ell^2 \eta_0^2 (u_r \underline{a}_r + i c_r^2 \eta_0) - u_\ell u_r c_\ell^2 |\check{\eta}|^2 (u_\ell \underline{a}_r + i c_r^2 \eta_0) \\ &= u_r \underline{a}_r c_\ell^2 (\eta_0^2 + u_\ell^2 |\check{\eta}|^2) - c_\ell^2 \eta_0^2 (u_r \underline{a}_r + i c_r^2 \eta_0) - u_\ell u_r c_\ell^2 |\check{\eta}|^2 (u_\ell \underline{a}_r + i c_r^2 \eta_0) \\ &= -i c_\ell^2 c_r^2 \eta_0 (\eta_0^2 + u_\ell u_r |\check{\eta}|^2). \end{aligned}$$

Similarly, the first factor in the expression (52) of  $A_r$  can be simplified by using

$$u_r \left\{ u_\ell \underline{a}_\ell (u_r \underline{a}_r + i c_r^2 \eta_0) \underline{\beta}_2^- + u_\ell c_r^2 |\check{\eta}|^2 (u_r \underline{a}_\ell + i c_\ell^2 \eta_0) \right\} = i c_\ell^2 c_r^2 \eta_0 (\eta_0^2 + u_\ell u_r |\check{\eta}|^2).$$

Using these two last simplifications, we can add (51) and (52) and obtain

$$(q_2 + q_3 + q_4)(k, k') = \frac{c_\ell^2}{\rho_\ell} Q_\ell + \frac{c_r^2}{\rho_r} Q_r + Q_\sharp,$$

with  $Q_\sharp$  defined in (33). This completes the proof of Proposition 2.  $\square$

We now compute the kernel  $q_2 + q_3 + q_4$  in the domain  $\{k > 0, k' > 0, k + k' > 0\}$ .

**Proposition 3.** *Let  $Q_\ell$  and  $Q_r$  be defined in (33) and let us define*

$$Q_b := -2 [\rho] [u] \underline{\Upsilon} (\eta_0^2 + u_r^2 |\check{\eta}|^2) u_\ell u_r |\check{\eta}|^2 \left\{ \frac{c_\ell^4 \underline{a}_r}{\rho_\ell \underline{a}_\ell} \overline{\gamma_1} (i \eta_0 - u_\ell \underline{\beta}_1^+) + \frac{c_r^4 \underline{a}_\ell}{\rho_r \underline{a}_r} \overline{\gamma_2} (i \eta_0 + u_r \underline{\beta}_2^+) \right\}.$$

Then the kernels  $q_2, q_3, q_4$  defined in (28), (29) and (30) satisfy

$$(q_2 + q_3 + q_4)(k, k') = \left\{ \left( \frac{p''(\rho_\ell)}{2} - \frac{c_\ell^2}{\rho_\ell} \right) \overline{Q_\ell} + \left( \frac{p''(\rho_r)}{2} - \frac{c_r^2}{\rho_r} \right) \overline{Q_r} + Q_b \right\} \left( 1 + \frac{k'}{k} \right),$$

for all  $(k, k')$  such that  $k > 0, k' < 0$  and  $k + k' > 0$ .

*Proof.* We split again the proof in several steps, as was done in the proof of Proposition 2.

• Step 1: computation of the  $p''(\rho_\ell)$  factor. We collect again the contributions that involve  $p''(\rho_\ell)$ . The contribution of the kernel  $q_2$  equals

$$\begin{aligned} & \underline{\Upsilon} (D_{d+1} + u_\ell D_{d+2}) |\gamma_1|^2 (-i \eta_0 + u_\ell \underline{\beta}_1^-) (i \eta_0 - u_\ell \underline{\beta}_1^+) \\ &= -[\rho] [u] \underline{\Upsilon} \overline{\gamma_1} |i \eta_0 - u_\ell \underline{\beta}_1^-|^2 \left\{ (\eta_0^2 + u_r^2 |\check{\eta}|^2) i c_r^2 \eta_0 + (u_r \underline{a}_r - i c_r^2 \eta_0) u_\ell u_r |\check{\eta}|^2 \right\}, \end{aligned} \quad (53)$$

where the product  $(D_{d+1} + u_\ell D_{d+2}) \gamma_1$  has already been computed when deriving (36).

The contribution of the kernel  $q_3$  equals

$$\begin{aligned}
& i(k+k') \int_0^{+\infty} \frac{\omega_1}{\gamma_1} (-i \check{\mathcal{J}}^T) e^{-(k+k') \underline{\beta}_1^+ z} |\gamma_1|^2 |i \mathcal{J}_0 - u_\ell \underline{\beta}_1^-|^2 \check{\mathcal{J}} e^{k \underline{\beta}_1^- z} e^{k' \underline{\beta}_1^+ z} dz \\
& + i(k+k') \int_0^{+\infty} \frac{\omega_3}{\gamma_1} (\mathcal{J}_0 \check{\mathcal{J}}^T) e^{-(k+k') \underline{\beta}_3^+ z} |\gamma_1|^2 |i \mathcal{J}_0 - u_\ell \underline{\beta}_1^-|^2 \check{\mathcal{J}} e^{k \underline{\beta}_1^- z} e^{k' \underline{\beta}_1^+ z} dz \\
& = \overline{\gamma_1} |i \mathcal{J}_0 - u_\ell \underline{\beta}_1^-|^2 \left\{ \left(1 + \frac{k'}{k}\right) \frac{\omega_1 |\check{\mathcal{J}}|^2}{\underline{\beta}_1^+ - \underline{\beta}_1^-} + \frac{i \mathcal{J}_0 (k+k') u_\ell |\check{\mathcal{J}}|^2 \omega_3}{k(i \mathcal{J}_0 - u_\ell \underline{\beta}_1^-) + k'(i \mathcal{J}_0 - u_\ell \underline{\beta}_1^+)} \right\}.
\end{aligned}$$

Similarly, the contribution of the kernel  $q_4$  reads

$$\begin{aligned}
& - \int_0^{+\infty} \frac{\omega_1}{\gamma_1} \underline{\beta}_1^+ e^{-(k+k') \underline{\beta}_1^+ z} |\gamma_1|^2 |i \mathcal{J}_0 - u_\ell \underline{\beta}_1^-|^2 (k \underline{\beta}_1^- + k' \underline{\beta}_1^+) e^{k \underline{\beta}_1^- z} e^{k' \underline{\beta}_1^+ z} dz \\
& - \int_0^{+\infty} \frac{\omega_3}{\gamma_1} u_\ell |\check{\mathcal{J}}|^2 e^{-(k+k') \underline{\beta}_3^+ z} |\gamma_1|^2 |i \mathcal{J}_0 - u_\ell \underline{\beta}_1^-|^2 (k \underline{\beta}_1^- + k' \underline{\beta}_1^+) e^{k \underline{\beta}_1^- z} e^{k' \underline{\beta}_1^+ z} dz \\
& = -\overline{\gamma_1} |i \mathcal{J}_0 - u_\ell \underline{\beta}_1^-|^2 \left\{ \frac{\omega_1}{\underline{\beta}_1^+ - \underline{\beta}_1^-} \left( \underline{\beta}_1^+ \underline{\beta}_1^- + \frac{k'}{k} (\underline{\beta}_1^+)^2 \right) + \frac{(k u_\ell \underline{\beta}_1^- + k' u_\ell \underline{\beta}_1^+) u_\ell |\check{\mathcal{J}}|^2 \omega_3}{k(i \mathcal{J}_0 - u_\ell \underline{\beta}_1^-) + k'(i \mathcal{J}_0 - u_\ell \underline{\beta}_1^+)} \right\}.
\end{aligned}$$

Adding the contributions of  $q_3$  and  $q_4$  gives the term

$$\overline{\gamma_1} |i \mathcal{J}_0 - u_\ell \underline{\beta}_1^-|^2 \left\{ -\frac{1}{2 \underline{a}_\ell} \left( \frac{\underline{a}_\ell^2}{c_\ell^2} + c_\ell^2 |\check{\mathcal{J}}|^2 \right) \omega_1 + u_\ell |\check{\mathcal{J}}|^2 \omega_3 + \frac{k'}{k} \omega_1 \frac{|\check{\mathcal{J}}|^2 - (\underline{\beta}_1^+)^2}{\underline{\beta}_1^+ - \underline{\beta}_1^-} \right\}. \quad (54)$$

We now add the contributions in (53) and (54) in order to obtain the  $p''(\rho_\ell)$  term in  $q_2 + q_3 + q_4$ . There is first a constant term that is independent of  $(k, k')$ , and this term is entirely similar to the one derived when adding (36) and (37) (see Step 1 in the proof of Proposition 2). Namely, the constant term in the sum of (53) and (54) equals

$$\begin{aligned}
& -[\rho][u] \underline{\Upsilon} \overline{\gamma_1} |i \mathcal{J}_0 - u_\ell \underline{\beta}_1^-|^2 \frac{\mathcal{J}_0^2 + u_r^2 |\check{\mathcal{J}}|^2}{2 c_\ell^2 (\mathcal{J}_0^2 + u_\ell^2 |\check{\mathcal{J}}|^2)} (u_\ell \underline{a}_\ell + i c_\ell^2 \mathcal{J}_0) \left( c_r^2 \mathcal{J}_0^2 + \frac{u_r \underline{a}_r}{u_\ell \underline{a}_\ell} u_\ell^2 c_\ell^2 |\check{\mathcal{J}}|^2 \right) \\
& = \frac{1}{2} [\rho][u] \underline{\Upsilon} \overline{\gamma_1} (-i \mathcal{J}_0 + u_\ell \underline{\beta}_1^+) (\mathcal{J}_0^2 + u_r^2 |\check{\mathcal{J}}|^2) \left( c_r^2 \mathcal{J}_0^2 + \frac{u_r \underline{a}_r}{u_\ell \underline{a}_\ell} u_\ell^2 c_\ell^2 |\check{\mathcal{J}}|^2 \right) = \frac{1}{2} \overline{Q}_\ell.
\end{aligned}$$

The last contribution in the  $p''(\rho_\ell)$  term is the one that depends on  $(k, k')$  in (54), that is

$$\begin{aligned}
& \frac{k'}{k} \overline{\gamma_1} |i \mathcal{J}_0 - u_\ell \underline{\beta}_1^-|^2 \omega_1 \frac{|\check{\mathcal{J}}|^2 - (\underline{\beta}_1^+)^2}{\underline{\beta}_1^+ - \underline{\beta}_1^-} \\
& = \frac{k'}{k} [\rho][u] \underline{\Upsilon} \overline{\gamma_1} |i \mathcal{J}_0 - u_\ell \underline{\beta}_1^-|^2 \frac{\mathcal{J}_0^2 + u_r^2 |\check{\mathcal{J}}|^2}{\mathcal{J}_0^2 + u_\ell^2 |\check{\mathcal{J}}|^2} \frac{(-u_\ell u_r \underline{a}_r)}{c_\ell^2} (u_\ell \underline{a}_\ell - i c_\ell^2 \mathcal{J}_0) \frac{(i \mathcal{J}_0 - u_\ell \underline{\beta}_1^+)^2}{2 c_\ell^2 \underline{a}_\ell} (c_\ell^2 - u_\ell^2) \\
& = \frac{k'}{2k} [\rho][u] \underline{\Upsilon} \overline{\gamma_1} (-i \mathcal{J}_0 + u_\ell \underline{\beta}_1^+) (\mathcal{J}_0^2 + u_r^2 |\check{\mathcal{J}}|^2) u_\ell u_r \frac{\underline{a}_r}{\underline{a}_\ell} \frac{|i \mathcal{J}_0 - u_\ell \underline{\beta}_1^-|^2}{c_\ell^2} (c_\ell^2 - u_\ell^2) = \frac{k'}{2k} \overline{Q}_\ell.
\end{aligned}$$

We have thus shown that the  $p''(\rho_\ell)$  term in  $(q_2 + q_3 + q_4)(k, k')$  is as claimed in Proposition 3.

• Step 2: computation of the  $p''(\rho_r)$  factor. The contribution of the kernel  $q_2$  equals

$$-\underline{\Upsilon} (D_{d+1} + u_r D_{d+2}) |\gamma_2|^2 |i \mathcal{J}_0 + u_r \underline{\beta}_2^-|^2 = [\rho][u] \underline{\Upsilon} \overline{\gamma_2} |i \mathcal{J}_0 + u_r \underline{\beta}_2^-|^2 \frac{u_\ell \underline{a}_\ell}{c_r^2} \left\{ i \mathcal{J}_0 u_r \underline{a}_r - u_r^2 c_r^2 |\check{\mathcal{J}}|^2 \right\}. \quad (55)$$

The contribution of the kernel  $q_3$  equals

$$\begin{aligned} i(k+k') \int_0^{+\infty} \frac{\omega_2}{\gamma_2} (i\check{\eta}^T) e^{-(k+k')\beta_2^+ z} |\gamma_2|^2 |i\eta_0 + u_r \beta_2^-|^2 \check{\eta} e^{k\beta_2^- z} e^{k'\beta_2^+ z} dz \\ = -\overline{\gamma_2} |i\eta_0 + u_r \beta_2^-|^2 \omega_2 \frac{|\check{\eta}|^2}{\beta_2^+ - \beta_2^-} \left(1 + \frac{k'}{k}\right), \end{aligned}$$

and the contribution of the kernel  $q_4$  reads

$$\begin{aligned} \int_0^{+\infty} \frac{\omega_2}{\gamma_2} \beta_2^+ e^{-(k+k')\beta_2^+ z} |\gamma_2|^2 |i\eta_0 + u_r \beta_2^-|^2 (k\beta_2^- + k'\beta_2^+) e^{k\beta_2^- z} e^{k'\beta_2^+ z} dz \\ = \overline{\gamma_2} |i\eta_0 + u_r \beta_2^-|^2 \frac{\omega_2}{\beta_2^+ - \beta_2^-} \left( \beta_2^+ \beta_2^- + \frac{k'}{k} (\beta_2^+)^2 \right). \end{aligned}$$

Adding the contributions of  $q_3$  and  $q_4$  gives the term

$$\overline{\gamma_2} |i\eta_0 + u_r \beta_2^-|^2 \left\{ -\frac{\omega_2}{2\underline{a}_r} \left( \frac{\underline{a}_r^2}{c_r^2} + c_r^2 |\check{\eta}|^2 \right) + \frac{k'}{k} \omega_2 \frac{(\beta_2^+)^2 - |\check{\eta}|^2}{\beta_2^+ - \beta_2^-} \right\}. \quad (56)$$

When we add the latter term with the expression in (55), we obtain the constant term (independent of  $(k, k')$ ):

$$-[\rho][u] \underline{\gamma} \overline{\gamma_2} |i\eta_0 + u_r \beta_2^-|^2 (\eta_0^2 + u_r^2 |\check{\eta}|^2) \frac{i u_r \eta_0}{2 \underline{a}_r} (u_\ell \underline{a}_\ell + i c_\ell^2 \eta_0) = \frac{1}{2} \overline{Q_r}.$$

The only term that depends on  $(k, k')$  arises in (56) and equals

$$\frac{k'}{k} \overline{\gamma_2} |i\eta_0 + u_r \beta_2^-|^2 \omega_2 \frac{(i\eta_0 + u_r \beta_2^+)^2}{2 c_r^2 \underline{a}_r} (c_r^2 - u_r^2) = \frac{k'}{2k} \overline{\gamma_2} (i\eta_0 + u_r \beta_2^+)^2 \frac{\eta_0^2 + u_r^2 |\check{\eta}|^2}{\underline{a}_r} \omega_2 = \frac{k'}{2k} \overline{Q_r}.$$

The sum of all the terms that involve  $p''(\rho_r)$  factorizes as claimed in Proposition 3.

• Step 3: computation of the remaining terms. In order to prove Proposition 3, we can assume from now on, and without loss of generality that  $p''(\rho_\ell) = p''(\rho_r) = 0$  in (34) and (35). With this simplification, we compute

$$\sum_{k=1}^{d-1} \check{\eta}_k d^2 f^k(v_\ell) \cdot (r_1^-, r_1^+) = -i \frac{c_\ell^4 |\check{\eta}|^2}{\rho_\ell} \begin{pmatrix} 0 \\ 2i\check{\eta} \\ -(\beta_1^+ + \beta_1^-) \end{pmatrix}, \quad (57)$$

$$d^2 \tilde{f}^d(v_\ell) \cdot (r_1^-, r_1^+) = \frac{c_\ell^4}{\rho_\ell} \begin{pmatrix} 0 \\ i(\beta_1^+ + \beta_1^-) \check{\eta} \\ -2\beta_1^+ \beta_1^- \\ -2u_\ell \beta_1^+ \beta_1^- \end{pmatrix}, \quad (58)$$

$$\sum_{k=1}^{d-1} \check{\eta}_k d^2 f^k(v_r) \cdot (r_2^-, r_2^+) = -i \frac{c_r^4 |\check{\eta}|^2}{\rho_r} \begin{pmatrix} 0 \\ 2i\check{\eta} \\ \beta_2^+ + \beta_2^- \end{pmatrix}, \quad (59)$$

$$d^2 \tilde{f}^d(v_r) \cdot (r_2^-, r_2^+) = -\frac{c_r^4}{\rho_r} \begin{pmatrix} 0 \\ i(\beta_2^+ + \beta_2^-) \check{\eta} \\ 2\beta_2^+ \beta_2^- \\ 2u_r \beta_2^+ \beta_2^- \end{pmatrix}, \quad (60)$$

As was done in Step 3 of the proof of Proposition 2, we are going to split the computations between the ‘left’ and ‘right’ contributions. Let us first concentrate on all ‘left’ terms. The ‘left’ contribution of  $q_2$  is computed from (58) and equals

$$\begin{aligned} \sigma^* d^2 \tilde{f}^d(v_\ell) \cdot (x_1^-, x_1^+) |\gamma_1|^2 &= \sigma^* \frac{c_\ell^4}{\rho_\ell} \begin{pmatrix} 0 \\ i(\underline{\beta}_1^+ + \underline{\beta}_1^-) \check{\eta} \\ -2 \underline{\beta}_1^+ \underline{\beta}_1^- \\ -2 u_\ell \underline{\beta}_1^+ \underline{\beta}_1^- \end{pmatrix} |\gamma_1|^2 \\ &= \frac{2 c_\ell^4}{\rho_\ell (c_\ell^2 - u_\ell^2)} \Upsilon \overline{\gamma_1} \left\{ u_\ell \eta_0 |\check{\eta}|^2 \gamma_1 \check{D} - (\eta_0^2 - c_\ell^2 |\check{\eta}|^2) \gamma_1 (D_{d+1} + u_\ell D_{d+2}) \right\}. \end{aligned}$$

Using Lemma 4, we find that the ‘left’ contribution of  $q_2$  is given by

$$\frac{2 c_\ell^4}{\rho_\ell (c_\ell^2 - u_\ell^2)} [\rho] [u] \Upsilon \overline{\gamma_1} \left\{ i c_r^2 \eta_0 (\eta_0^2 - c_\ell^2 |\check{\eta}|^2) (\eta_0^2 + u_r^2 |\check{\eta}|^2) - u_\ell u_r c_\ell^2 |\check{\eta}|^4 (u_r \underline{a}_r - i c_r^2 \eta_0) \right\}. \quad (61)$$

Similarly, we use (60) to compute the ‘right’ contribution of  $q_2$ , which yields

$$\begin{aligned} -\sigma^* d^2 \tilde{f}^d(v_\ell) \cdot (x_2^-, x_2^+) |\gamma_2|^2 &= \frac{2 c_r^4}{\rho_r (c_r^2 - u_r^2)} [\rho] [u] \Upsilon \overline{\gamma_2} \\ &\quad \times \left\{ (i c_\ell^2 \eta_0^3 + u_\ell \underline{a}_\ell u_r^2 |\check{\eta}|^2) (\eta_0^2 - c_r^2 |\check{\eta}|^2) - \eta_0^2 u_r^2 |\check{\eta}|^2 (u_\ell \underline{a}_\ell - i c_\ell^2 \eta_0) \right\}. \end{aligned} \quad (62)$$

Using (57), the ‘left’ contribution of  $q_3$  equals

$$\begin{aligned} &i(k+k') \int_0^{+\infty} \frac{\omega_1}{\gamma_1} \tilde{\ell}_1 e^{-(k+k')} \underline{\beta}_1^+ z \frac{-i c_\ell^4 |\check{\eta}|^2}{\rho_\ell} \begin{pmatrix} 0 \\ 2i \check{\eta} \\ -(\underline{\beta}_1^+ + \underline{\beta}_1^-) \end{pmatrix} |\gamma_1|^2 e^k \underline{\beta}_1^- z e^{k'} \underline{\beta}_1^+ z dz \\ &+ i(k+k') \int_0^{+\infty} \frac{\omega_3}{\gamma_1} \tilde{\ell}_3 e^{-(k+k')} \underline{\beta}_3^+ z \frac{-i c_\ell^4 |\check{\eta}|^2}{\rho_\ell} \begin{pmatrix} 0 \\ 2i \check{\eta} \\ -(\underline{\beta}_1^+ + \underline{\beta}_1^-) \end{pmatrix} |\gamma_1|^2 e^k \underline{\beta}_1^- z e^{k'} \underline{\beta}_1^+ z dz \\ &= \frac{c_\ell^4}{\rho_\ell} \overline{\gamma_1} \omega_1 \tilde{\ell}_1 \begin{pmatrix} 0 \\ 2i \check{\eta} \\ -(\underline{\beta}_1^+ + \underline{\beta}_1^-) \end{pmatrix} \frac{|\check{\eta}|^2}{\underline{\beta}_1^+ - \underline{\beta}_1^-} \left(1 + \frac{k'}{k}\right) \\ &\quad + \underbrace{\frac{c_\ell^4}{\rho_\ell} \overline{\gamma_1} \omega_3 \tilde{\ell}_1 \begin{pmatrix} 0 \\ 2i \check{\eta} \\ -(\underline{\beta}_1^+ + \underline{\beta}_1^-) \end{pmatrix} \frac{(k+k') |\check{\eta}|^2}{k(\underline{\beta}_3^+ - \underline{\beta}_1^-) + k'(\underline{\beta}_3^+ - \underline{\beta}_1^+)}}_{\diamond}, \end{aligned} \quad (63)$$

and, similarly (using now (59) rather than (57)), the ‘right’ contribution of  $q_3$  equals

$$\begin{aligned} &i(k+k') \int_0^{+\infty} \frac{\omega_2}{\gamma_2} \tilde{\ell}_2 e^{-(k+k')} \underline{\beta}_2^+ z \frac{-i c_r^4 |\check{\eta}|^2}{\rho_r} \begin{pmatrix} 0 \\ 2i \check{\eta} \\ \underline{\beta}_2^+ + \underline{\beta}_2^- \end{pmatrix} |\gamma_2|^2 e^k \underline{\beta}_2^- z e^{k'} \underline{\beta}_2^+ z dz \\ &= \frac{c_r^4}{\rho_r} \overline{\gamma_2} \omega_2 \tilde{\ell}_2 \begin{pmatrix} 0 \\ 2i \check{\eta} \\ \underline{\beta}_2^+ + \underline{\beta}_2^- \end{pmatrix} \frac{|\check{\eta}|^2}{\underline{\beta}_2^+ - \underline{\beta}_2^-} \left(1 + \frac{k'}{k}\right). \end{aligned} \quad (64)$$

The ‘left’ contribution of  $q_4$  is computed by retaining the three first rows in (58):

$$\begin{aligned}
& - \int_0^{+\infty} \frac{\omega_1}{\gamma_1} \tilde{\ell}_1 e^{-(k+k')} \underline{\beta}_1^+ z \frac{c_\ell^4}{\rho_\ell} \begin{pmatrix} 0 \\ i(\underline{\beta}_1^+ + \underline{\beta}_1^-) \check{\mathcal{J}} \\ -2 \underline{\beta}_1^+ \underline{\beta}_1^- \end{pmatrix} |\gamma_1|^2 (k \underline{\beta}_1^- + k' \underline{\beta}_1^+) e^{k \underline{\beta}_1^- z} e^{k' \underline{\beta}_1^+ z} dz \\
& - \int_0^{+\infty} \frac{\omega_3}{\gamma_1} \tilde{\ell}_1 e^{-(k+k')} \underline{\beta}_3^+ z \frac{c_\ell^4}{\rho_\ell} \begin{pmatrix} 0 \\ i(\underline{\beta}_1^+ + \underline{\beta}_1^-) \check{\mathcal{J}} \\ -2 \underline{\beta}_1^+ \underline{\beta}_1^- \end{pmatrix} |\gamma_1|^2 (k \underline{\beta}_1^- + k' \underline{\beta}_1^+) e^{k \underline{\beta}_1^- z} e^{k' \underline{\beta}_1^+ z} dz \\
& = - \frac{c_\ell^4}{\rho_\ell} \overline{\gamma_1} \omega_1 \tilde{\ell}_1 \begin{pmatrix} 0 \\ i(\underline{\beta}_1^+ + \underline{\beta}_1^-) \check{\mathcal{J}} \\ -2 \underline{\beta}_1^+ \underline{\beta}_1^- \end{pmatrix} \frac{1}{\underline{\beta}_1^+ - \underline{\beta}_1^-} \left( \underline{\beta}_1^- + \frac{k'}{k} \underline{\beta}_1^+ \right) \\
& \quad - \underbrace{\frac{c_\ell^4}{\rho_\ell} \overline{\gamma_1} \omega_3 \tilde{\ell}_1 \begin{pmatrix} 0 \\ i(\underline{\beta}_1^+ + \underline{\beta}_1^-) \check{\mathcal{J}} \\ -2 \underline{\beta}_1^+ \underline{\beta}_1^- \end{pmatrix} \frac{k \underline{\beta}_1^- + k' \underline{\beta}_1^+}{k(\underline{\beta}_3^+ - \underline{\beta}_1^-) + k'(\underline{\beta}_3^+ - \underline{\beta}_1^+)}}_{\diamond}, \tag{65}
\end{aligned}$$

and the ‘right’ contribution of  $q_4$  equals

$$\begin{aligned}
& - \frac{c_r^4}{\rho_r} \overline{\gamma_2} \omega_2 \tilde{\ell}_2 \begin{pmatrix} 0 \\ i(\underline{\beta}_2^+ + \underline{\beta}_2^-) \check{\mathcal{J}} \\ 2 \underline{\beta}_2^+ \underline{\beta}_2^- \end{pmatrix} \frac{1}{\underline{\beta}_2^+ - \underline{\beta}_2^-} \left( \underline{\beta}_2^- + \frac{k'}{k} \underline{\beta}_2^+ \right) \\
& = \frac{c_r^4}{\rho_r} \overline{\gamma_2} \omega_2 \tilde{\ell}_2 \begin{pmatrix} 0 \\ i(\underline{\beta}_2^+ + \underline{\beta}_2^-) \check{\mathcal{J}} \\ 2 \underline{\beta}_2^+ \underline{\beta}_2^- \end{pmatrix} - \frac{c_r^4}{\rho_r} \overline{\gamma_2} \omega_2 \tilde{\ell}_2 \begin{pmatrix} 0 \\ i(\underline{\beta}_2^+ + \underline{\beta}_2^-) \check{\mathcal{J}} \\ 2 \underline{\beta}_2^+ \underline{\beta}_2^- \end{pmatrix} \frac{\underline{\beta}_2^+}{\underline{\beta}_2^+ - \underline{\beta}_2^-} \left( 1 + \frac{k'}{k} \right). \tag{66}
\end{aligned}$$

The ‘left’ contribution of  $q_3 + q_4$  is computed as follows. We first observe that the difference between the diamond term in (63) and the diamond term in (65) reads

$$\frac{c_\ell^2}{\rho_\ell} \overline{\gamma_1} \omega_3 u_\ell |\check{\mathcal{J}}|^2 \left\{ c_\ell^2 |\check{\mathcal{J}}|^2 - c_\ell^2 \underline{\beta}_1^+ \underline{\beta}_1^- - (i \underline{\eta}_0 - u_\ell \underline{\beta}_1^+) (i \underline{\eta}_0 - u_\ell \underline{\beta}_1^-) \right\} = \frac{2 c_\ell^6}{\rho_\ell (c_\ell^2 - u_\ell^2)} u_\ell |\check{\mathcal{J}}|^4 \overline{\gamma_1} \omega_3.$$

Consequently the ‘left’ contribution of  $q_3 + q_4$ , which corresponds to the sum of (63) and (65), equals

$$\begin{aligned}
& \frac{c_\ell^4 \overline{\gamma_1} \omega_1}{\rho_\ell (\underline{\beta}_1^+ - \underline{\beta}_1^-)} \left\{ |\check{\mathcal{J}}|^2 \tilde{\ell}_1 \begin{pmatrix} 0 \\ 2i \check{\mathcal{J}} \\ -(\underline{\beta}_1^+ + \underline{\beta}_1^-) \end{pmatrix} - \underline{\beta}_1^+ \tilde{\ell}_1 \begin{pmatrix} 0 \\ i(\underline{\beta}_1^+ + \underline{\beta}_1^-) \check{\mathcal{J}} \\ -2 \underline{\beta}_1^+ \underline{\beta}_1^- \end{pmatrix} \right\} \left( 1 + \frac{k'}{k} \right) \\
& + \frac{c_\ell^4}{\rho_\ell} \overline{\gamma_1} \omega_1 \tilde{\ell}_1 \begin{pmatrix} 0 \\ i(\underline{\beta}_1^+ + \underline{\beta}_1^-) \check{\mathcal{J}} \\ -2 \underline{\beta}_1^+ \underline{\beta}_1^- \end{pmatrix} + \frac{2 c_\ell^6}{\rho_\ell (c_\ell^2 - u_\ell^2)} u_\ell |\check{\mathcal{J}}|^4 \overline{\gamma_1} \omega_3 \\
& = \frac{2 c_\ell^4}{\rho_\ell (\underline{\beta}_1^+ - \underline{\beta}_1^-)} \overline{\gamma_1} \omega_1 (\underline{\beta}_1^+ \underline{\beta}_1^- - |\check{\mathcal{J}}|^2) ((\underline{\beta}_1^+)^2 - |\check{\mathcal{J}}|^2) \left( 1 + \frac{k'}{k} \right) \\
& \quad - \frac{2 c_\ell^4}{\rho_\ell (c_\ell^2 - u_\ell^2)} \overline{\gamma_1} \omega_1 \left\{ (\underline{\eta}_0^2 - c_\ell^2 |\check{\mathcal{J}}|^2) \underline{\beta}_1^+ + i u_\ell \underline{\eta}_0 |\check{\mathcal{J}}|^2 \right\} + \frac{2 c_\ell^6}{\rho_\ell (c_\ell^2 - u_\ell^2)} u_\ell |\check{\mathcal{J}}|^4 \overline{\gamma_1} \omega_3.
\end{aligned}$$

After a little bit of algebra, the ‘left’ contribution of  $q_3 + q_4$  is found to be equal to

$$\begin{aligned}
& -\frac{c_\ell^2}{\rho_\ell} [\rho] [u] \underline{\Upsilon} u_\ell u_r \frac{\underline{a}_r}{\underline{a}_\ell} (\varpi_0^2 + u_r^2 |\check{\varpi}|^2) \left( \frac{\underline{a}_\ell^2}{c_\ell^2} + c_\ell^2 |\check{\varpi}|^2 \right) \overline{\gamma}_1 (i \varpi_0 - u_\ell \underline{\beta}_1^+) \left( 1 + \frac{k'}{k} \right) \\
& + \frac{2 c_\ell^4}{\rho_\ell (c_\ell^2 - u_\ell^2)} [\rho] [u] \underline{\Upsilon} \overline{\gamma}_1 \left\{ -i c_r^2 \varpi_0 (\varpi_0^2 - c_\ell^2 |\check{\varpi}|^2) (\varpi_0^2 + u_r^2 |\check{\varpi}|^2) + u_\ell u_r c_\ell^2 |\check{\varpi}|^4 (u_r \underline{a}_r - i c_r^2 \varpi_0) \right\}.
\end{aligned}$$

When combined with (61), we have thus shown that the ‘left’ contribution of the kernel  $q_2 + q_3 + q_4$  reads

$$\begin{aligned}
& -\frac{c_\ell^2}{\rho_\ell} [\rho] [u] \underline{\Upsilon} u_\ell u_r \frac{\underline{a}_r}{\underline{a}_\ell} (\varpi_0^2 + u_r^2 |\check{\varpi}|^2) \left( \frac{\underline{a}_\ell^2}{c_\ell^2} + c_\ell^2 |\check{\varpi}|^2 \right) \overline{\gamma}_1 (i \varpi_0 - u_\ell \underline{\beta}_1^+) \left( 1 + \frac{k'}{k} \right) \\
& = -\left( \frac{c_\ell^2}{\rho_\ell} \overline{Q}_\ell + 2 [\rho] [u] \underline{\Upsilon} u_\ell u_r |\check{\varpi}|^2 (\varpi_0^2 + u_r^2 |\check{\varpi}|^2) \frac{c_\ell^4 \underline{a}_r}{\rho_\ell \underline{a}_\ell} \overline{\gamma}_1 (i \varpi_0 - u_\ell \underline{\beta}_1^+) \right) \left( 1 + \frac{k'}{k} \right). \quad (67)
\end{aligned}$$

Let us now compute the ‘right’ contribution of the kernel  $q_3 + q_4$ . We add the expressions in (64) and (66), which gives

$$\frac{2 c_r^4}{\rho_r} \overline{\gamma}_2 \omega_2 \left\{ (\underline{\beta}_2^+ \underline{\beta}_2^-) \underline{\beta}_2^+ - \frac{\underline{\beta}_2^+ + \underline{\beta}_2^-}{2} |\check{\varpi}|^2 \right\} - \frac{2 c_r^4}{\rho_r} \overline{\gamma}_2 \omega_2 ((\underline{\beta}_2^+)^2 - |\check{\varpi}|^2) \frac{\underline{\beta}_2^+ \underline{\beta}_2^- - |\check{\varpi}|^2}{\underline{\beta}_2^+ - \underline{\beta}_2^-} \left( 1 + \frac{k'}{k} \right),$$

or equivalently

$$\begin{aligned}
& \frac{2 c_r^4}{\rho_r (c_r^2 - u_r^2)} [\rho] [u] \underline{\Upsilon} \overline{\gamma}_2 i u_r \varpi_0 (u_\ell \underline{a}_\ell - i c_\ell^2 \varpi_0) \left\{ (\varpi_0^2 - c_r^2 |\check{\varpi}|^2) \underline{\beta}_2^+ - i u_r \varpi_0 |\check{\varpi}|^2 \right\} \\
& - \frac{c_r^4}{\rho_r} [\rho] [u] \underline{\Upsilon} u_\ell u_r \frac{\underline{a}_\ell}{\underline{a}_r} (\varpi_0^2 + u_r^2 |\check{\varpi}|^2) \left( \frac{\underline{a}_r^2}{c_r^2} + c_r^2 |\check{\varpi}|^2 \right) \overline{\gamma}_2 (i \varpi_0 + u_r \underline{\beta}_2^+) \left( 1 + \frac{k'}{k} \right). \quad (68)
\end{aligned}$$

The first row in (68) is exactly the opposite of (62), that is of the ‘right’ contribution of  $q_2$ . In other words, we have found that the ‘right’ contribution of  $q_2 + q_3 + q_4$  is given by the second row in (68), which reads

$$-\left( \frac{c_r^2}{\rho_r} \overline{Q}_r + 2 [\rho] [u] \underline{\Upsilon} u_\ell u_r |\check{\varpi}|^2 (\varpi_0^2 + u_r^2 |\check{\varpi}|^2) \frac{c_r^4 \underline{a}_\ell}{\rho_r \underline{a}_r} \overline{\gamma}_2 (i \varpi_0 + u_r \underline{\beta}_2^+) \right) \left( 1 + \frac{k'}{k} \right).$$

The kernel  $q_2 + q_3 + q_4$  is the sum of the latter quantity and the right hand side of (67). Collecting the terms, this completes the proof of Proposition 3.  $\square$

**Corollary 2.** *With  $Q$  defined in (32),  $Q_\ell, Q_r, Q_\sharp, Q_b$  defined in Propositions 2 and 3, the kernel  $q$  satisfies*

$$q(k, k') = \begin{cases} \left( \left( \frac{p''(\rho_\ell)}{2} + \frac{c_\ell^2}{\rho_\ell} \right) Q_\ell + \left( \frac{p''(\rho_r)}{2} + \frac{c_r^2}{\rho_r} \right) Q_r + Q_\sharp \right) & \text{if } k > 0 \text{ and } k' > 0, \\ \left\{ \left( \frac{p''(\rho_\ell)}{2} - \frac{c_\ell^2}{\rho_\ell} \right) \overline{Q}_\ell + \left( \frac{p''(\rho_r)}{2} - \frac{c_r^2}{\rho_r} \right) \overline{Q}_r + Q_b + \overline{Q} \right\} \left( 1 + \frac{k'}{k} \right) & \text{if } k > 0, k' < 0, k + k' > 0. \end{cases}$$

## 4 Conclusion

There remains a final simplification in order to achieve the final form of the kernel  $q$ . Our main result reads as follows.

**Proposition 4.** *With  $Q_\ell, Q_r, Q_\#$  defined in Propositions 2, let us define*

$$Q_{\natural} := \left( \frac{p''(\rho_\ell)}{2} + \frac{c_\ell^2}{\rho_\ell} \right) Q_\ell + \left( \frac{p''(\rho_r)}{2} + \frac{c_r^2}{\rho_r} \right) Q_r + Q_\#.$$

*Then the kernel  $q = 4\pi a_1$  satisfies*

$$q(k, k') = \begin{cases} Q_{\natural} & \text{if } k > 0 \text{ and } k' > 0, \\ \frac{Q_{\natural}}{Q_{\natural}} \left( 1 + \frac{k'}{k} \right) & \text{if } k > 0, k' < 0 \text{ and } k + k' > 0. \end{cases}$$

*In particular,  $q$  satisfies Hunter's stability condition  $q(1, 0^+) = \overline{q(1, 0^-)}$ , and (15) is locally well-posed in  $H^2(\mathbb{R})$ .*

*Proof.* In view of the expression of  $q$  given in Corollary 2, we only need to show the relation

$$\frac{c_\ell^2}{\rho_\ell} Q_\ell + \frac{c_r^2}{\rho_r} Q_r + \frac{1}{2} Q_\# = \frac{1}{2} (Q + \overline{Q_b}),$$

with  $Q_\ell, Q_r, Q_\#$  given in Proposition 2,  $Q_b$  in Proposition 3, and  $Q$  in Lemma 1. Simplifying by the common factor  $[\rho][u]\Upsilon(\varpi_0^2 + u_\ell^2 |\check{\varpi}|^2) u_\ell u_r$ , we are led to showing that the following relation holds:

$$\begin{aligned} & \frac{\underline{a}_r}{\rho_\ell \underline{a}_\ell} c_\ell^2 (\varpi_0^2 + u_\ell^2 |\check{\varpi}|^2) \gamma_1 (i \varpi_0 - u_\ell \underline{\beta}_1^-) + \frac{\underline{a}_\ell}{\rho_r \underline{a}_r} c_r^2 (\varpi_0^2 + u_r^2 |\check{\varpi}|^2) \gamma_2 (i \varpi_0 + u_r \underline{\beta}_2^-) \\ & - \frac{1}{\mathbf{j}[u] \varpi_0} (\varpi_0^2 + u_\ell u_r |\check{\varpi}|^2) i \underline{a}_\ell \underline{a}_r (c_r^2 \gamma_2 - c_\ell^2 \gamma_1) = (\underline{\beta}_1^- + \underline{\beta}_2^-) i \underline{a}_\ell \underline{a}_r \frac{u_\ell \underline{a}_r + i c_r^2 \varpi_0}{u_\ell \underline{a}_r - i c_r^2 \varpi_0} \\ & + \frac{\underline{a}_r}{\rho_\ell \underline{a}_\ell} c_\ell^4 |\check{\varpi}|^2 \gamma_1 (i \varpi_0 - u_\ell \underline{\beta}_1^-) + \frac{\underline{a}_\ell}{\rho_r \underline{a}_r} c_r^4 |\check{\varpi}|^2 \gamma_2 (i \varpi_0 + u_r \underline{\beta}_2^-), \quad (69) \end{aligned}$$

where we use the notation  $\mathbf{j} := \rho_\ell u_\ell = \rho_r u_r$  to denote the mass flux across the phase boundary.

The verification of (69) proceeds as follows. We first combine the first and third row in (69) by recalling the definitions of  $\underline{a}_\ell, \underline{a}_r$ , see (1) and (2). Thus verifying (69) amounts to showing

$$\begin{aligned} & (\underline{\beta}_1^- + \underline{\beta}_2^-) \frac{u_\ell \underline{a}_r + i c_r^2 \varpi_0}{u_\ell \underline{a}_r - i c_r^2 \varpi_0} - i \frac{\gamma_1}{\rho_\ell} (i \varpi_0 - u_\ell \underline{\beta}_1^-) - i \frac{\gamma_2}{\rho_r} (i \varpi_0 + u_r \underline{\beta}_2^-) \\ & + \frac{1}{\mathbf{j}[u] \varpi_0} (\varpi_0^2 + u_\ell u_r |\check{\varpi}|^2) (c_r^2 \gamma_2 - c_\ell^2 \gamma_1) = 0. \quad (70) \end{aligned}$$

Let us now define

$$\begin{aligned} B_\ell &:= \underline{\beta}_1^- \frac{u_\ell \underline{a}_r + i c_r^2 \varpi_0}{u_\ell \underline{a}_r - i c_r^2 \varpi_0} - i \frac{\gamma_1}{\rho_\ell} (i \varpi_0 - u_\ell \underline{\beta}_1^-) - \frac{1}{\mathbf{j}[u] \varpi_0} (\varpi_0^2 + u_\ell u_r |\check{\varpi}|^2) c_\ell^2 \gamma_1, \\ B_r &:= \underline{\beta}_2^- \frac{u_\ell \underline{a}_r + i c_r^2 \varpi_0}{u_\ell \underline{a}_r - i c_r^2 \varpi_0} - i \frac{\gamma_2}{\rho_r} (i \varpi_0 + u_r \underline{\beta}_2^-) + \frac{1}{\mathbf{j}[u] \varpi_0} (\varpi_0^2 + u_\ell u_r |\check{\varpi}|^2) c_r^2 \gamma_2, \end{aligned}$$

so that (70), which is the relation we wish to prove, reads  $B_\ell + B_r = 0$ .

Using the relations

$$\frac{u_\ell \underline{a}_r + i c_r^2 \eta_0}{u_\ell \underline{a}_r - i c_r^2 \eta_0} = -\frac{u_r \underline{a}_\ell + i c_\ell^2 \eta_0}{u_r \underline{a}_\ell - i c_\ell^2 \eta_0}, \quad (u_r \underline{a}_\ell - i c_\ell^2 \eta_0) \gamma_1 = -\rho_\ell [u] \eta_0,$$

we get

$$u_\ell (u_r \underline{a}_\ell - i c_\ell^2 \eta_0) B_\ell = -i \eta_0 (u_\ell \underline{a}_\ell + i c_\ell^2 \eta_0). \quad (71)$$

Similarly, we compute

$$u_r (u_\ell \underline{a}_r - i c_r^2 \eta_0) B_r = -i \eta_0 (u_r \underline{a}_r + i c_r^2 \eta_0). \quad (72)$$

Combining the relations (71) and (72), we get

$$\begin{aligned} & u_\ell u_r (u_r \underline{a}_\ell - i c_\ell^2 \eta_0) (u_\ell \underline{a}_r - i c_r^2 \eta_0) (B_\ell + B_r) \\ &= -i \eta_0 u_r (u_\ell \underline{a}_r - i c_r^2 \eta_0) (u_\ell \underline{a}_\ell + i c_\ell^2 \eta_0) - i \eta_0 u_\ell (u_r \underline{a}_\ell - i c_\ell^2 \eta_0) (u_r \underline{a}_r + i c_r^2 \eta_0) \\ &= -i \eta_0 (u_\ell + u_r) (u_\ell u_r \underline{a}_\ell \underline{a}_r + c_\ell^2 c_r^2 \eta_0^2) = 0. \end{aligned}$$

This means that (70) holds, and consequently the expression of the kernel  $q$  is as claimed in Proposition 4. The verification of Hunter's stability condition  $q(1, 0^+) = \overline{q(1, 0^-)}$  is then straightforward, and local well-posedness in  $H^2(\mathbb{R})$  follows from the main result in [2].  $\square$

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